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Preface

These notes contain a systematized description of the contents of my series of talks held at Kyoto University in January, 1968. Principal theorems of which only outlines were announced in the talks are given here proofs; they are some of my recent results which have not yet been published elsewhere.

T. Kubota,

January 22, 1969.

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Introduction

In the present paper, we shall intend to describe a few basic steps in the study of a territory of the number theory which possibly includes a satisfactory generalization of the remarkable relationship between automorphic forms and the reciprocity law pointed out by Hecke in the last chapter of his book [5].

This kind of phenomenon appears rather topological and real analytic than algebraic or complex analytic. In fact, a clear, theoretical explanation of the result of Hecke can be given as a simple application by the general theory of [23] concerning unitary operators. The investigation in this paper has also a similar chracter.

We shall first observe the general linear group G_F of degree 2 over a totally imaginary number field F containing the n-th roots of unity for some fixed $n \geq 2$, and construct an n-fold, topological covering group $\tilde{G}_A$ of the adele group G_A of G_F through an explicit factor set. In the case of $n = 2$, $\tilde{G}_A$ coincides essentially with a metaplectic group in the sense of [23]. While the construction of the covering group with arbitrary n was already done by [15] and [9], the explicit factor set used in this paper enables us to deal rapidly with various concrete problems.

Next, we shall consider an arithmetical discontinuous group Γ built up from G_F which acts, completely analogously to Hilbert's modular group, on a direct product H^r of the real, three dimensional upper half space H, that is, a non-hermitian symmetric space called usually quaternian hyperbolic space. If, under this situation, f is a function on H^r satisfying $f(\sigma u) = \chi(\sigma) f(u)$, $(\sigma \in \Gamma)$, where χ is a representation of the type introduced in the theorem of [7], then f corresponds in a natural way to a function on

$\tilde{G}_A$. We shall construct a finite dimensional Hilbert space Θ over $\mathbb{C}$ of such functions, which are in addition square integrable on $\Gamma \backslash H^r$, by using real analytic Eisenstein series $E(u, s, \chi)$ in the sense of [17], [18] at a singularity with respect to the complex parameter s. This procedure has its origine in a general remark in [18] that as the residue at a pole of an Eisenstein series one gets an automorphic form which is square integrable on a fundamental domain, but is not a cusp form.

It is one of the principal conclusions in this paper that a function f as above can be regarded as a function on $\tilde{G}_A$, but not on G_A (Prop. 5); this is so to speak another expression of the fact that, as is shown in [1], the existence of a non-congruence subgroup of finite index of Γ is equivalent to the existence of a non-trivial covering of G_A.

The notion of the Hecke ring can also be extended to the covering group $\tilde{G}_A$. Indeed, the Hecke ring of $\tilde{G}_A$ can be defined in the frame of the general theory of automorphic functions, and it is almost commutative in the sense that Theorem 4 holds. Furthermore, the finite dimensional space Θ_χ is mapped into itself by the operation of the Hecke ring (Theorem 8). Thus, by the general theory, Θ_χ is decomposed into a sum of irreducible subspaces with respect to the operation of the Hecke ring, and each direct component in the decomposition determines an irreducible unitary representation of $\tilde{G}_A$ in a space of automorphic forms. In our case, however, we can show moreover that the representation is of a simple and explicit nature. For instance, the zonal spherical function of the representation is expressed by using an arithmetical sum.

§ 1. GL(2) over a totally imaginary field.

Throughout the paper, we denote by F a totally imaginary number field of finite degree which contains the n-th roots of unity for a fixed $n \geqq 2$. If $n > 2$, then the assumption that F is totally imaginary is automatically satisfied. We denote by G_F the general linear group $GL(2, F)$ of degree 2 over F, and we put $G_{\mathfrak{p}} = GL(2, F_{\mathfrak{p}})$ for a prime $\mathfrak{p}$ of F, where $F_{\mathfrak{p}}$ means the $\mathfrak{p}$-adic completion of F. If $\mathfrak{p}$ is finite, and if $\mathfrak{o}_{\mathfrak{p}}$ is the ring of integers of $F_{\mathfrak{p}}$, then, for any matric group G over $\mathfrak{o}_{\mathfrak{p}}$ and for a natural number N, we denote by G_N the congruence subgroup mod. N of G, i. e. the group of all $\sigma \in G$ with $\sigma \equiv I \pmod{N}$, where I is the identity matrix. The congruence subgroup of a matric group over the ring $\mathfrak{o}$ of integers of F is defined similarly.

The adele group of G_F with usual topology will be denoted by G_A. For the $\mathfrak{p}$-component of an adele a, we use the notation $a_{\mathfrak{p}}$, $(a)_{\mathfrak{p}}$, or $pr_{\mathfrak{p}} a$, and these will sometimes be identified with the adele of which the $\mathfrak{p}$-component equals to that of a and all other local components are 1. The product of all $pr_{\mathfrak{p}} a$ for finite $\mathfrak{p}$ will be denoted by $pr_0 a$, and will be called the finite component of a. The infinite component $pr_\infty a$ of a is $\prod_{\mathfrak{p}|\infty} pr_{\mathfrak{p}} a$. We put $G_0 = pr_0 G_A$, $G_\infty = pr_\infty G_A$, and more generally we write $X_0 = pr_0 X$, and $X_\infty = pr_\infty X$ for any subset X of G_A. On the other hand, we put $K_{\mathfrak{p}} = GL(2, \mathfrak{o}_{\mathfrak{p}})_N$ for finite $\mathfrak{p}$, and $K_{\mathfrak{p}} = U(2)$ for infinite $\mathfrak{p}$, then $K = \prod_{\mathfrak{p}} K_{\mathfrak{p}}$ is a compact subgroup of G_A, and the $\mathfrak{p}$-component of $a \in G_A$ belongs to $K_{\mathfrak{p}}$ for almost all $\mathfrak{p}$. We denote furthermore by Δ the group of all adeles a such that $a_{\mathfrak{p}}$ is diagonal for all infinite $\mathfrak{p}$, and 1 for all finite $\mathfrak{p}$.

The group of principal adeles will be identified with G_F. G_F is then a discrete subgroup of G_A, and the number h' of double cosets $G_F g K_0 G_\infty$, $(g \in G_A)$, is finite [2]. So, there exist $a_i \in G_0$, $(i = 1, 2, \dots, h')$ such that every double coset $G_F g K_0 G_\infty$ is of the form $G_F a_i K_0 G_\infty$ for some i, and $G_F K_0 G_\infty$ is a normal subgroup of index h' of G_A. If $N = 1$, then h' is equal to the class number of F.

The space $G_F \backslash G_A / \Delta K$ has a finite volume with respect to a natural measure induced by the Haar measure, but is not compact [2]. Let f be a function on this space, and let a_i be as above. Then, through the relation

$$(1) \qquad f(g) = f_i(g_\infty) \quad \text{for} \quad g = \xi a_i \cdot k g_\infty, \quad (\xi \in G_F,\ k \in K_0,\ g_\infty \in G_\infty),$$

we have functions f_i on G_∞ with the property

$$(2) \qquad f_i(\sigma g_\infty \omega) = f_i(g_\infty)$$

for $\omega \in \Delta K_\infty$, $\sigma \in pr_\infty(a_i^{-1} G_F a_i \cap K_0 G_\infty) = pr_\infty(G_F \cap a_i K_0 G_\infty a_i^{-1})$.

Conversely, if we have functions f_i satisfying (2) for all i, then (1) determines a function f on $G_F \backslash G_A / \Delta K$. Thus, putting $\Gamma_{a_i} = pr_\infty(G_F \cap a_i K_0 G_\infty a_i^{-1})$, we have a one to one correspondence between functions on $G_F \backslash G_A / \Delta K$ and vectors of functions f_i on $\Gamma_{a_i} \backslash G_\infty / \Delta K_\infty$. In order that f is square integrable, it is necessary and sufficient that all f_i are square integrable with respect to a natural measure. If $a_i = 1$, then we write Γ for Γ_{a_i}; Γ is equal to $GL(2, \mathfrak{o})_N$.

Now, if $(F:\mathbf{Q}) = 2r$, then $G_\infty / \Delta K_\infty$ is the direct product H^r of r copies of the quaternion hyperbolic space $H = SL(2, \mathbf{C})/SU(2)$. Some elementary properties of the space H are summarized in [10]. In particular, H is realized as a real, three dimensioanl upper-half space, if we denote by $u = \begin{pmatrix} z & -v \\ v & \bar{z} \end{pmatrix}$, $(z = x + \sqrt{-1}\, y \in C,\ v > 0)$, a point in the upper-half space and define the operation $u \to \sigma u$, $(\sigma \in SL(2, \mathbf{C}))$, by

$$(3) \qquad \sigma u = (\tilde{a}u + \tilde{b})(\tilde{c}u + \tilde{d})^{-1},$$

where $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and $\tilde{w} = \begin{pmatrix} w & \\ & \bar{w} \end{pmatrix}$ for any $w \in \mathbf{C}$. Since (3) has a well-defined meaning even if $v \leq 0$, we can regard $u \to \sigma u$ in (3) as a transformation of $\mathbf{R}^3$ with coordinates x, y, v. If we add to $\mathbf{R}^3$ a point ∞ at infinity, and if we identify the (x, y)-plane with the complex plane $\mathbf{C}$, then $\mathbf{C}$ is invariant under σ. A natural left invariant metric on H can also be given explicitly by

$$(4) \qquad ds^2 = \frac{1}{v^2}(dx^2 + dy^2 + dv^2).$$

To the direct product $H^r = G_\infty / \Delta K_\infty$, we give the product metric of (4). It is of course G_∞-left invariant and there are r independent Laplacians on H^r. While the operation of G_∞ on H^r is induced by (3) in a natural way, it should be noted that the operation of $\sigma \in \Gamma_{a_i}$ on H^r is of "Hilbert's type". Namely, if $u = (u_1, \ldots, u_r) \in H^r = G_\infty / \Delta K_\infty$, then

$$(5) \qquad \sigma u = (\sigma^{(1)} u_1, \ldots, \sigma^{(r)} u_r),$$

where $\sigma^{(\ell)}$ is the ℓ-th conjugate of σ over $\mathbf{Q}$ in the usual sense of algebraic number theory. For any $\sigma \in GL(2, F)$, the transformation in (5) represents also a transformation of the direct product of r copies of $\mathbf{R}^3$. If we identify F with the set of points $\gamma = (\gamma^{(1)}, \ldots, \gamma^{(r)}) \in \mathbf{C} \times \cdots \times \mathbf{C}$, and if we denote by ∞ a symbolical point at infinity[1)], then the set $F \cup \{\infty\}$ is invariant under $\sigma \in GL(2, F)$.

At least geometrically and group theoretically, there is no essential difference between our discontinuous group Γ_{a_i} and Hilbert's modular group. In particular, we can construct a fundamental domain $\mathcal{D}_i = \Gamma_{a_i} \backslash H^r$ in a quite similar way to the case of Hilbert's modular group. On the other hand, we can classify all elements in $GL(2, F)$ according to the action on H^r. To do this, note first that every $\sigma \in SL(2, \mathbf{C})$, $(\sigma \neq 1)$, is classified into four classes which are called elliptic, hyperbolic, loxodromic, and parabolic [10]. Now, if some conjugate $\sigma^{(\ell)}$ of $\sigma \in GL(2, F)$ is elliptic, then all other conjugates of σ are also elliptic. Because, in order to $\sigma^{(\ell)}$ is elliptic, it is necessary and sufficient that the characteristic polynomial of $\sigma^{(\ell)}$ is of the form $x^2 + \sqrt{a}\,(\zeta + \zeta^{-1})x + a$, where $a = \det \sigma^{(\ell)}$, and ζ is a root of unity different from ± 1. So, in this case, we say that σ is elliptic. Considering the case of $\zeta = \pm 1$, we see also that all conjugates of σ are parabolic whenever one $\sigma^{(\ell)}$ is so; in this case we call σ parabolic. The situation is not quite similar for hyperbolic and loxodromic cases. Namely, it is possible that some $\sigma^{(\ell)}$ is hyperbolic, while another $\sigma^{(\ell)}$ is loxodromic. Therefore, we shall use in

1) This point at infinity differs from the above-mentioned one added to $\mathbf{R}^3$.

this case such terminologies as mixed, totally hyperbolic, etc.

If $\sigma \in GL(2, F)$ is parabolic, then its two characteristic roots are equal, and belong to F. Therefore, there exists a $\tau \in SL(2, F)$ such that $\tau^{-1}\sigma\tau$ is of the form $\begin{pmatrix} a & b \\ & a \end{pmatrix}$. From this it follows in particular that σ has a unique fixed point in the set $F \cup \{\infty\}$. Now, let Γ' be a subgroup of finite index of our discontinuous group Γ_{a_i}, and call the fixed point of a parabolic $\sigma \in \Gamma'$ a cusp of Γ'. Then, the above remark implies that the set of all cusps of Γ' is exactly the set $F \cup \{\infty\}$, and that the group of all elements of Γ' which leave a cusp fixed contains a normal subgroup isomorphic to $\mathbf{Z}^{2r}$ consisting of all parabolic elements in the group. The number of Γ_{a_i}-inequivalent cusps of Γ_{a_i} is finite; in particular it is equal to the class number of F when $N = 1$. Furthermore, we can show that, if a boundary point of a fundamental domain of Γ' does not lie in H^r, then it is a cusp of Γ'. These are all analogies of corresponding results in the theory of Hilbert's modular group.

§ 2. Hecke operators and automorphic functions.

For our purpose in this paper, it is not adequate to study automorphic functions with respect to the group G_A, the adele group of $GL(2, F)$, because what we need actually is automorphic functions over a certain covering group of G_A, and not on G_A itself. So, to help the later description, we state here some generalities about automorphic function on a topological group. Almost all facts stated in this § are well-known[2)], our intention is merely to gather them together in a convenient form for use in the sequel.

Let G be a locally compact, unimodular group whose Haar measure is denoted by dx or μ, and let Γ [resp. K] be a discrete [resp. compact] subgroup of G such that the measure $\mu(K) = \int_K dx$ is 1. We then consider functions f on G satisfying

$$(6) \qquad f(\gamma x) = f(x), \qquad \gamma \in \Gamma,$$

and

$$(7) \qquad f(xk) = f(x)\chi(k), \qquad k \in K,$$

where $\chi(k)$ is a character (representation of degree 1) of K. Such a function will be called an automorphic function on G with respect to Γ, K and χ.

Let ψ be a continuous, complex valued function with compact support on G, and let f be a function satisfying (6) and (7), i. e. an automorphic

2) See among others [3], [6], [16], and [21].

function. Denote by

$$(f_1 * f_2)(x) = \int_G f_1(xy^{-1}) f_2(y)\, dy$$

the convolution of two functions f_1, f_2 on G. Then, in general, $f * \psi$ does not belong to the space of automorphic functions, while

$$(8) \quad f^{T(\psi)}(x) = f(x) \circ T(\psi) = \int_K (f * \psi)(xk^{-1}) \chi(k)\, dk = \int_K \int_G f(xk^{-1}y^{-1}) \psi(y) \chi(k)\, dy dk$$

does. The operator $T(\psi) : f \to f^{T(\psi)} = f \circ T(\psi)$ will be called the Hecke operator determined by ψ. If the kernel K_1 of the representation χ is open, and ψ is $\mu(K_1)^{-1}$-times the characteristic function of the set $K_1 a$, $(a \in G)$, then we write $T(a)$ for $T(\psi)$. In this case, we have

$$(9) \qquad f^{T(a)}(x) = \int_K f(xk^{-1}a^{-1}) \chi(k)\, dk.$$

Two functions $\psi(y)$ and $\psi(k_0^{-1} y k_0)$, $(k_0 \in K)$, determine one and the same Hecke operator. This shows $T(\psi) = T(\psi^o)$ with

$$(10) \qquad \psi^o(x) = \int_K \psi(k^{-1}xk) dk.$$

Let now $T(\psi_1)$, $T(\psi_2)$ be two Hecke operators with $\psi_1^o = \psi_1$, $\psi_2^o = \psi_2$, and define the composition $T(\psi_1) \cdot T(\psi_2)$ by $f^{T(\psi_1) \cdot T(\psi_2)} = (f^{T(\psi_1)})^{T(\psi_2)}$. Then we have $T(\psi_1) \cdot T(\psi_2) = T(\psi_1 * \psi_2)$. Furthermore, if we define

$T(\psi_1) + T(\psi_2)$ by $f^{T(\psi_1)+T(\psi_2)} = f^{T(\psi_1)} + f^{T(\psi_2)}$, and $\alpha T(\psi)$, $(\alpha \in \mathbf{C})$, by $f^{\alpha T(\psi)} = \alpha f^{T(\psi)}$, then $T(\psi_1) + T(\psi_2) = T(\psi_1 + \psi_2)$, and $\alpha T(\psi) = T(\alpha\psi)$. Therefore, all Hecke operators form an algebra over $\mathbf{C}$.

If $\psi^o = \psi$, then it follows from (8) that

$$f^{T(\psi)}(x) = \int_G f(xy^{-1}) \int_K \psi(yk^{-1})\chi(k)dkdy.$$

So, if we put

$$\psi_\chi(x) = \int_K \psi(xk^{-1})\chi(k)dk,$$

then $f^{T(\psi)} = f^{T(\psi_\chi)} = f * \psi_\chi$, and ψ_χ has the property

$$\psi_\chi(kxk') = \chi(k)\psi_\chi(x)\chi(k') \tag{11}$$

for any $k, k' \in K$. All continuous functions satisfying (11) with compact support on G form an algebra $\mathfrak{ae}_\chi(G, K)$ over $\mathbf{C}$, which we shall call the Hecke ring (algebra) with respect to G, K, and χ. The Hecke operators with a fixed Γ constitute a representation of the Hecke algebra ; if $f^{T(\psi_1)} = f * \psi_{1,\chi}$, $f^{T(\psi_2)} = f * \psi_{2,\chi}$, then $f^{T(\psi_1)T(\psi_2)} = f * (\psi_{1,\chi} * \psi_{2,\chi})$. If K is not open in G, the algebra $\mathfrak{ae}(G, K)$ contains no unit element. But, even then, we can add a unit element to the algebra, and the unit element is realized in a natural way as the limit of some sequence of operators in the previous $\mathfrak{ae}(G, K)$. So, from now on, we always understand that the Hecke algebra $\mathfrak{a}(G, K)$ is completed to contain a unit element.

We assume now that the measure of the quotient space $\Gamma\backslash G$ is finite, and for two functions f_1, f_2 on $\Gamma\backslash G$, we define an inner product by

$$(f_1, f_2) = \int_{\Gamma\backslash G} f_1(x)\overline{f_2(x)}dx.$$

This defines a structure of Hilbert space on $\mathfrak{H} = L^2(\Gamma\backslash G)$, the space of all complex valued functions f on $\Gamma\backslash G$ with $\|f\|^2 = (f, f) < \infty$ containing all constant functions. If we put $(U_g f)(x) = f(xg)$ for $g \in G$ and $f \in \mathfrak{H}$, then U_g is a unitary operator. On the other hand, the operator P defined by

$$(12) \qquad (Pf)(x) = \int_K f(xk^{-1})\chi(k)dk = (\int_K \chi(k)U_k^{-1}dk)f$$

is a projection of $\mathfrak{H}$, and the Hecke operator in (8) is expressed as

$$(13) \qquad T(\psi) = P\int_G \psi(y)U_y^{-1}dy.$$

We recall here a fundamental connection between Hecke operators and the unitary representation. Let M be a finite dimensional subspace of $\mathfrak{H}$ over $\mathbf{C}$ consisting of automorphic functions on G with respect to Γ, K and χ, and assume $M^{T(\psi)} \subset M$ for all Hecke operators. Furthermore, X being a subset of $\mathfrak{H}$, denote by $\mathfrak{H}_X$ the closed subspace of $\mathfrak{H}$ generated by all $U_g f$, $(f \in X, g \in G)$. Then, since $P[f(xg)] = PU_g f$, it follows from (13) and from the assumption that $PU_g f \in M$ for $f \in M$. This means $P\mathfrak{H}_M = M$ because of $PM = M$. Now, let M_1 be an irreducible subspace of M under the operation of the Hecke algebra $æ_\chi(G, K)$. The orthogonal complement $\mathfrak{H}_1$ of $\mathfrak{H}_{M_1}$ is invariant under U_g, $(g \in G)$. Besides, we have $P\mathfrak{H}_1 \subset \mathfrak{H}_1$, because P is self-adjoint by (12). Therefore, $P\mathfrak{H}_1 = M_1' \subset M$ must be the orthogonal complement of M_1 in M. Since the adjoint operator of $T(\psi)$ is $T(\overline{\psi}(x^{-1})) \in æ_\chi(G, K)$, $M_1'^{\,T(\psi)}$ is orthogonal to M_1, and is contained in M_1'. Thus, $\mathfrak{H}_M$ is the direct, orthogonal sum of $\mathfrak{H}_{M_1}$, where $M = \oplus M_i$ is a decomposition of M into the orthogonal, direct sum of irreducible subspaces with respect to $æ_\chi(G, K)$, and each M_i has the property $PM_i = M_i$. Another important fact is that the unitary representation of G determined by the restriction of U_g to M_i is irreducible. To see this, take the union $\mathfrak{H}'$ of all those $\{U_g\}$-invariant subspaces of $\mathfrak{H}_{M_i}$ whose projections by P are 0, and let $\mathfrak{H}''$ be the orthogonal complement in $\mathfrak{H}_{M_i}$ of $\mathfrak{H}'$. Then, since $\mathfrak{H}''$ is $\{U_g\}$-invariant, every non-zero $f \in \mathfrak{H}''$ has the property $PU_{g'}f \neq 0$ for some $g' \in G$, and $PU_{g'}f$ belongs to M_i. Therefore, it follows from the irreducibility of M_i that $\mathfrak{H}_{\{f\}} = \mathfrak{H}_{M_i}$. This proves $\mathfrak{H}'' = \mathfrak{H}_{M_i}$, and at the same time our assertion.

We now propose to prepare some concrete formulas on the operation of Hecke operators. We shall mainly consider the case where the Hecke algebra $æ_\chi(G, K)$ is defined by a locally compact, unimodular group G with a character χ of a compact subgroup K of G such that the kernel K_1 of χ is open; we treat the operator $T(a)$ defined by (9) for $a \in G$. Let f be an automorphic function, satisfying (6), (7) by definition, and let $K = \bigcup (K \cap a^{-1}Ka)\sigma_i$, $K \cap a^{-1}Ka = \bigcup (K_1 \cap a^{-1}K_1 a)\tau_j$ be coset decompositions. Then,

$K = \bigcup_{i,j} (K_1 \quad a^{-1}K_1 a)\tau_j\sigma_i$ is also a coset decomposition, and it follows from (9) that

$$f^{T(a)}(x) = \mu(K_1 \cap a^{-1}K_1 a) \sum_{i,j} f(x\sigma_i^{-1}\tau_j^{-1}a^{-1})\chi(\tau_j\sigma_i).$$

So, we obtain

$$(14) \qquad f^{T(a)}(x) = c_\chi(a) \sum_i f(x\sigma_i^{-1}a^{-1})\chi(\sigma_i)$$

with

$$c_\chi(a) = \mu(K_1 \cap a^{-1}K_1 a) \sum_j \chi(a\tau_j^{-1}a^{-1}\tau_j).$$

It should be noted that the coset decomposition $KaK = \bigcup Ka\sigma_i$ holds with our σ_i.

We here restrict our observation to the case of the adele group G_A of the general linear group $G_F = GL(2, F)$ over a totally imaginary number field. Notations being as in § 1, let f_A be a function on $G_F \backslash G_F K_0 G_\infty / \Delta$ satisfying $f_A(gk) = f_A(g)\chi(k)$ for $k \in K$, where χ is a character of K. Then, f_A determines a function f on the direct product H^r of r copies of the upper-half space H such that

$$(15) \qquad f(\gamma u) = \chi(pr_0(\gamma^{-1}))f(u), \qquad (\gamma \in \Gamma).$$

The correspondence between f_A and f is given by

$$(16) \qquad f(g_\infty) = f_A(g) \qquad \text{for} \qquad g = \xi k g_\infty$$

with $\xi \in G_F$, $k \in K$, $g_\infty \in G_\infty$.

Every result which was obtained in this § for a general topological group G applies of course for the case where we use $G = G_F K_0 G_\infty / \Delta$ together with its discrete and compact subgroups G_F and $K\Delta/\Delta$. This case will be basic in the sequel. We shall, however, often abbreviate Δ for the sake of simplicity. For example, the Hecke ring $æ_\chi(G, K\Delta/\Delta)$ will be denoted by $æ_\chi(G, K)$ for $G = G_F K_0 G_\infty/\Delta$. If the character χ is trivial, then we write $æ(G, K)$ for $æ_\chi(G, K)$.

We propose here to observe $æ(G, K)$, and to get a formula which gives $f^{T(a)}$, $(a \in G_0,\ pr_{\mathfrak{p}}\, a = 1$ for $\mathfrak{p} | N)$, determined by $f^{T(a)}(g_\infty) = f_A^{T(a)}(g)$[3]. In this case, there exists an $\alpha \in G_F$ such that $K_0 a K_0 = K_0 (\alpha)_0 K_0$, and it follows from the local theory of elementary divisors and from the approximation theorem for $SL(2)$ that the double cosets $K_0(\alpha)_0 K_0$ are in one to one natural correspondence with double cosets $\Gamma(\alpha)_\infty \Gamma$. So, if $\Gamma(\alpha)_\infty\Gamma = \cup \Gamma\alpha_i$ is a coset decomposition, then one obtains

$$(17) \qquad (K ; K \cap a^{-1} K a) f^{T(a)}(u) = \sum f(\alpha_i u)$$

using (14) and (16).

---

3) By (9), $T(a)$ is defined as an element of $æ(G_0, K_0)$, but it has a well defined meaning also as an element of $æ(G, K)$, because $æ(G_\infty, K_\infty)$ is completed to contain a unit element, and $æ(G, K)$ is the tensor product of $æ(G_\infty, K_\infty)$ and $æ(G_0, K_0)$.

§ 3. Construction of a covering group.

Previously we investigated the adele group G_A of $GL(2, F)$ over a totally imaginary number field F containig the n-th roots of unity for a fixed $n \geqq 2$. We are now going to construct an n-fold, topological covering group $\widetilde{G}_A$ of G_A by means of an explicit factor set, and to exhibit some fundamental properties of $\widetilde{G}_A$. Such a group was discovered in [23] for the first time for $n = 2$, and was called a metaplectic group. So, even in case $n \geqq 2$, we shall use the same name. The construction of the metaplectic group in the general case was done independently by [15], and the metaplectic group has an intimate connection with those subgroups of $GL(2, \mathfrak{o})$ which contain no congruence subgroup[4], $\mathfrak{o}$ being the ring of integers of F.

Theorem 1. Let $\mathfrak{p}$ be a place of F, and put $G_{\mathfrak{p}} = GL(2, F_{\mathfrak{p}})$. For $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, F_{\mathfrak{p}})$, define $x(\sigma)$ by

$$x(\sigma) = \begin{cases} c, & c \neq 0, \\ d, & c = 0, \end{cases}$$

and put

$$(18) \qquad a(\sigma, \tau) = (x(\sigma), x(\tau))(-x(\sigma)^{-1}x(\tau), x(\sigma\tau))$$

for $\sigma, \tau \in SL(2, F_{\mathfrak{p}})$, where $(*, *)$ means the Hilbert-Hasse's norm residue

4) See [1], [14], and [19].

symbol of degree n of $F_{\mathfrak{p}}$. Furthermore, for a $\sigma \in GL(2, F_{\mathfrak{p}})$, denote by $p(\sigma)$ the element of $SL(2, F_p)$ determined by $\sigma = \begin{pmatrix} 1 & \\ & \det\sigma \end{pmatrix} p(\sigma)$, denote by σ^y, $(y \in F_{\mathfrak{p}},\ y \neq 0)$, the matrix $\begin{pmatrix} 1 & \\ & y \end{pmatrix}^{-1} \sigma \begin{pmatrix} 1 & \\ & y \end{pmatrix}$, and put

$$v(y, \sigma) = \begin{cases} 1, & c \neq 0, \\ (y, d), & c = 0, \end{cases} \qquad \left(\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right).$$

Then,

$$(19) \qquad a(\sigma, \tau) = a(p(\sigma)^{\det\tau}, p(\tau))v(\det\tau, p(\sigma)), \qquad \sigma, \tau \in G_{\mathfrak{p}},$$

is a factor set which determines a topological covering group $\tilde{G}_{\mathfrak{p}}$ of $G_{\mathfrak{p}}$ such that $\tilde{G}_{\mathfrak{p}}$ is central as a group extension.

Proof. It was proved in [9] that (18) determines a topological covering $\widetilde{SL}(2, F_{\mathfrak{p}})$ of $SL(2, F_{\mathfrak{p}})$ which is central as a group extension. Let $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\tau = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ be two elements of $SL(2, F_{\mathfrak{p}})$, and put $\sigma\tau = \begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix}$. Then, for any non-zero $y \in F_{\mathfrak{p}}$, we have

$$(20) \qquad a(\sigma, \tau) = a(\sigma^y, \tau^y)v(y, \sigma)v(y, \tau)v(y, \sigma\tau)^{-1}.$$

This can be proved by simple, direct computations. Indeed, if $c \neq 0$, $c' \neq 0$, and $c'' = 0$, then

$$\begin{aligned} a(\sigma, \tau)a(\sigma^y, \tau^y)^{-1} &= (c, c')(-c^{-1}c', d'')(cy^{-1}, c'y^{-1})^{-1}(-c^{-1}c', d'')^{-1} \\ &= (y, -c^{-1}c') = (y, a'') = (y, d'')^{-1}, \end{aligned}$$

if $c = 0,\ c' \neq 0$, then

$$\begin{aligned} a(\sigma, \tau)a(\sigma^y, \tau^y)^{-1} &= (d, c')(-d^{-1}c', dc')(d, c'y^{-1})^{-1}(-d^{-1}c'y^{-1}, dc'y^{-1})^{-1} \\ &= (y, d), \end{aligned}$$

and if $c \neq 0,\ c' = 0$, then

$$\begin{aligned} a(\sigma, \tau)a(\sigma^y, \tau^y)^{-1} &= (c, d')(-c^{-1}d', d'^{-1}c)(cy^{-1}, d')^{-1}(-c^{-1}d'y, d'^{-1}cy^{-1})^{-1} \\ &= (y, d') . \end{aligned}$$

other cases are obvious.

We denote an element of $\widetilde{SL}(2, F_{\mathfrak{p}})$ by the symbol $(\sigma, \zeta)_a$, $(\sigma \in SL(2, F_{\mathfrak{p}}), \zeta^n = 1)$, so that the multiplication in $\widetilde{SL}(2, F_{\mathfrak{p}})$ is given by $(\sigma, \zeta_\sigma)_a(\tau, \zeta_\tau)_a = (\sigma\tau, \zeta_\sigma\zeta_\tau a(\sigma, \tau))_a$. Now, let $\tilde{\sigma} = (\sigma, \zeta)_a$ be an element of $SL(2, F_{\mathfrak{p}})$ and put $\tilde{\sigma}^y = (\sigma^y, \zeta v(y, \sigma))_a$. Then, it follows from (20) that $\tilde{\sigma} \rightarrow \tilde{\sigma}^y$, is an automorphism of $\widetilde{SL}(2, F_{\mathfrak{p}})$, and from the definition follows $(\tilde{\sigma}^y)^{y'} = \tilde{\sigma}^{yy'}$ for two non-zero $y, y' \in F_{\mathfrak{p}}$. Therefore

$$\left[\begin{pmatrix}1 & \\ & y\end{pmatrix}, \tilde{\sigma}\right]\left[\begin{pmatrix}1 & \\ & y'\end{pmatrix}, \tilde{\sigma}'\right] = \left[\begin{pmatrix}1 & \\ & yy'\end{pmatrix}, \tilde{\sigma}^{y'}\tilde{\sigma}'\right]$$

defines a structure of topological group on the set $\tilde{G}_{\mathfrak{p}}$ of all pairs $\left[\begin{pmatrix}1 & \\ & y\end{pmatrix}, \tilde{\sigma}\right]$, and through the mapping $\left[\begin{pmatrix}1 & \\ & y\end{pmatrix}, \tilde{\sigma}\right] \rightarrow \begin{pmatrix}1 & \\ & y\end{pmatrix}\sigma$, $(\tilde{\sigma} = (\sigma, \zeta)_a)$,

$\tilde{G}_{\mathfrak{p}}$ becomes a covering group of $G_{\mathfrak{p}}$ satisfying all conditions in the theorem.

Theorem 1 assures the existence of a covering over the local group $G_{\mathfrak{p}}$. The next theorem, which explains the behaviour of the factor set $a(\sigma, \tau)$ on the compact subgroup $K_{\mathfrak{p}}$ of $G_{\mathfrak{p}}$ defined in § 1, is useful in constructing a global covering of the adele group G_A.

Theorem 2. Let $\mathfrak{p}$ be a finite prime of F, and let N be a natural number divisible by n^2; then the factor set $a(\sigma, \tau)$ in (19) splits on the compact subgroup $K_{\mathfrak{p}}$ of $G_{\mathfrak{p}}$. More precisely, we have

$$a(\sigma, \tau) = s(\sigma)s(\tau)s(\sigma\tau)^{-1}, \qquad (\sigma, \tau \in K_{\mathfrak{p}}),$$

with

$$s(\sigma) = \begin{cases} (c, d \det \sigma^{-1})^{-1}, & \text{if } cd \text{ is not zero, and } \operatorname{ord}_{\mathfrak{p}} c \text{ is} \\ & \text{not divisible by } n, \\ 1, & \text{otherwise,} \end{cases}$$

for $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_{\mathfrak{p}}$.

Proof. Since σ, τ are in $K_{\mathfrak{p}}$, the above definition of $s(\sigma)$ is equivalent to

$$s(\sigma) = \begin{cases} (c, d \det \sigma^{-1})^{-1}, & \text{if } c \text{ is neither } 0 \text{ nor a unit,} \\ 1, & \text{otherwise.} \end{cases}$$

The proof will be performed by using this modified definition, because it is more convenient than the original, while the original definition is necessary for the later application.

First we assume $\sigma, \tau \in SL(2, \mathfrak{o}_{\mathfrak{p}})$, and put $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\tau = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$, $\sigma\tau = \begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix}$. If c, c', or c'' is zero, then the validity of the theorem can be chequed by a direct computation using fundamental properties of norm residue symbol[5]. Namely, if $c = 0$, $c' \neq 0$, and if c' not a unit, then $a(\sigma, \tau) = (c', d)$, while $s(\sigma) = 1$, $s(\tau) = (c', d')^{-1}$, and $s(\sigma\tau) = (c', dd')^{-1}$. If $c \neq 0$ is not a unit, and $c' = 0$, then $a(\sigma, \tau) = (c, d')$, while $s(\sigma) = (c, d)^{-1}$, $s(\tau) = 1$, and $s(\sigma\tau) = (cd'^{-1}, b'c + dd')^{-1} = (c, dd')^{-1}$. If $c \neq 0$ is not a unit and $c'' = 0$, then $\tau = \begin{pmatrix} * & * \\ -d''^{-1}c & -cb'' + ad'' \end{pmatrix}$, and therefore $a(\sigma, \tau) = (c, d''^{-1})$, while $s(\sigma) = (c, d)^{-1}$, $s(\tau) = (-d''^{-1}c, -cb'' + ad'')^{-1} = (c, dd''^{-1})$, and $s(\sigma\tau) = 1$. Other cases with $cc'c'' = 0$ are trivial. So, we may assume now $cc'c'' \neq 0$. If $\nu \in SL(2, F_{\mathfrak{p}})$ is of the form $\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$, then it follows from (18) that $a(\nu\sigma, \tau) = a(\sigma\nu^{-1}, \nu\tau) = a(\sigma, \tau\nu)$. On the other hand, if such a ν belongs to $SL(2, \mathfrak{o}_{\mathfrak{p}})_N$, then $s(\nu\sigma) = s(\sigma\nu) = s(\sigma)$. This means that we may replace σ, τ by $\nu_1\sigma\nu_2$, $\nu_2^{-1}\sigma\nu_3$ to prove the theorem, where ν_1, ν_2, ν_3 are in $SL(2, \mathfrak{o}_{\mathfrak{p}})_N$ and of the form $\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$. Therefore, unless both c and c' are non-units, the proof of the theorem reduces to the following three cases:

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5) A convenient formula for our purpose here is the formula (1) of [7].

$$\text{i)}\quad \sigma = \begin{pmatrix} & -\varepsilon^{-1} \\ \varepsilon & \end{pmatrix}, \qquad \tau = \begin{pmatrix} a' & -\varepsilon'^{-1} \\ \varepsilon' & \end{pmatrix}, \qquad \sigma\tau = \begin{pmatrix} -\varepsilon^{-1}\varepsilon' & \\ a'\varepsilon & -\varepsilon\varepsilon'^{-1} \end{pmatrix},$$

$$\text{ii)}\quad \sigma = \begin{pmatrix} \varepsilon & \\ c & \varepsilon^{-1} \end{pmatrix}, \qquad \tau = \begin{pmatrix} & -\varepsilon'^{-1} \\ \varepsilon' & \end{pmatrix}, \qquad \sigma\tau = \begin{pmatrix} & -\varepsilon\varepsilon'^{-1} \\ \varepsilon^{-1}\varepsilon' & -c\varepsilon'^{-1} \end{pmatrix},$$

$$\text{iii)}\quad \sigma = \begin{pmatrix} & -\varepsilon^{-1} \\ \varepsilon & \end{pmatrix}, \qquad \tau = \begin{pmatrix} \varepsilon' & \\ c' & \varepsilon'^{-1} \end{pmatrix}, \qquad \sigma\tau = \begin{pmatrix} -c'\varepsilon^{-1} & -\varepsilon^{-1}\varepsilon'^{-1} \\ \varepsilon\varepsilon' & \end{pmatrix}.$$

Here, ε, ε' are units, and c, c' are non-units. If a' is zero or a unit, then i) becomes trivial. Excluding this case, the proof of the theorem for i), ii), iii) can again be carried out by an elementary calculation. Thus the only remaining case is the one of

$$\sigma = \begin{pmatrix} \varepsilon & \\ c & \varepsilon^{-1} \end{pmatrix}, \qquad \tau = \begin{pmatrix} \varepsilon' & \\ c' & \varepsilon'^{-1} \end{pmatrix}, \qquad \sigma\tau = \begin{pmatrix} \varepsilon\varepsilon' & \\ c\varepsilon' + c'\varepsilon^{-1}, & \varepsilon^{-1}\varepsilon'^{-1} \end{pmatrix},$$

where ε, ε' are units, and non of c, c' is a unit. In this case, we have

$$\begin{aligned} & a(\sigma, \tau)s(\sigma)^{-1}s(\tau)^{-1}s(\sigma\tau) \\ &= (c, c')(-c^{-1}c', c\varepsilon' + c'\varepsilon^{-1})(c, \varepsilon)^{-1}(c', \varepsilon')^{-1}(c\varepsilon' + c'\varepsilon^{-1}, \varepsilon\varepsilon') \\ &= (c\varepsilon', c'\varepsilon^{-1})(-(c\varepsilon')^{-1}c'\varepsilon^{-1}, c\varepsilon' + c'\varepsilon^{-1}). \end{aligned}$$

But this is 1 because of a well-known property[6] of the norm residue symbol.

We now turn to the general case, and observe the factor set (19) for σ, $\tau \in K_{\mathfrak{p}}$. On one hand, we have $a(\sigma, \tau) = a(p(\sigma)^{\det \tau}, p(\tau))$ $= s(p(\sigma)^{\det \tau})s(p(\tau))s(p(\sigma)^{\det \tau}p(\tau))^{-1}$, using the above results for $SL(2, \mathfrak{o}_{\mathfrak{p}})$.

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6) The formula (1) in [9].

On the other hand, $s(p(\sigma)^{\det\tau}) = s(p(\sigma))$, $p(\sigma)^{\det\tau}p(\tau) = p(\sigma\tau)$, and, for any $\sigma \in K_{\mathfrak{p}}$, $s(p(\sigma)) = s(\sigma)$ by definition. Hence, the theorem is proved.

Theorem 2 was proved under the assumption that $\sigma, \tau \in GL(2, \mathfrak{o}_{\mathfrak{p}})$ and that $\mathfrak{p}$ does not divide N. But the number $s(\sigma)$ in the theorem is well-defined even if σ is an arbitrary element of $G_{\mathfrak{p}}$, or if $\mathfrak{p}|N$. So, we define a new factor set $b(\sigma, \tau)$ of $G_{\mathfrak{p}}$ by

$$(21) \qquad b(\sigma, \tau) = a(\sigma, \tau)s(\sigma)^{-1}s(\tau)^{-1}s(\sigma\tau), \qquad (\sigma, \tau \in G_{\mathfrak{p}}),$$

for an arbitrary $\mathfrak{p}$[7]. The assertion of Theorem 2 is nothing but $b(\sigma, \tau) = 1$ for $\sigma, \tau \in K_{\mathfrak{p}}$, when $\mathfrak{p}$ does not divide N.

Let now g, g' be two adeles in G_A; then $b(g_{\mathfrak{p}}, g'_{\mathfrak{p}})$, $a(g_{\mathfrak{p}}, g'_{\mathfrak{p}})$, and $s(g_{\mathfrak{p}})$ are all well-defined. These will be denoted by $b_{\mathfrak{p}}(g, g')$, $a_{\mathfrak{p}}(g, g')$, and $s_{\mathfrak{p}}(g)$, respectively. Since $b_{\mathfrak{p}}(g, g') = 1$ for almost all $\mathfrak{p}$, we can define a factor set b_A of G_A by

$$(22) \qquad b_A(g, g') = \prod_{\mathfrak{p}} b_{\mathfrak{p}}(g, g'), \qquad (g, g' \in G_A),$$

the product being extended over all places of F. The factor set g_A determines a central group extension $\tilde{G}_A$ of G_A. Namely, $\tilde{G}_A$ is realized as the set of all pairs (g, ζ), $(g \in G_A, \zeta^n = 1)$, with the group opetation defined by

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7) If $\mathfrak{p}$ is infinite, we put always $s(\sigma) = 1$. In this case both $a(\sigma, \tau)$ and $b(\sigma, \tau)$ are trivial.

(23) $\qquad (g, \zeta)(g', \zeta') = (gg', b_A(g, g')\zeta\zeta')$

between two such pairs. We denote the element $(1, \zeta) \in \widetilde{G}_A$ by $\dot{\zeta}$. The group $\mathfrak{z}$ of all $\dot{\zeta}$ is contained in the center of $\widetilde{G}_A$, and $\dot{\zeta} \leftrightarrow \zeta$ gives an isomorphism between $\mathfrak{z}$ and the group of the n-th roots of unity in F.

Let N be a natural number divisible by n^2, and let K be the compact subgroup of G_A defined in § 1. Then, it follows from Theorem 2 that $K \ni k \to (k, 1) \in G_A$ is a group-theoretical isomorphism. Whenever no confusion is possible, we identify the image of the above mapping with K, and denote $(k, 1)$ simply by k. Through this identification $K \subset \widetilde{G}_A$ is given a structrue of a compact topological group, and the topology coincides on $K_\mathfrak{p} \subset \widetilde{G}_\mathfrak{p}$ with the previous covering topology of $\widetilde{G}_\mathfrak{p}$ because $s(\sigma)$ in Theorem 2 vanishes on a suitable neighbourhood in $G_\mathfrak{p}$ of 1. On the other hand, the set of all $(g, 1)$, $(g \in G_\infty)$, forms a group isomorphic to G_∞. Identifying $(g, 1)$ with g, we have a subgroup $G_\infty \subset \widetilde{G}_A$, to which we give the same topological structure as $G_\infty \subset G_A$. It should be noted here that the subsets of the groups K, G_∞ will also be identified with corresponding subsets of $\widetilde{G}_A$. If we require now that $\widetilde{G}_A / K_0 G_\infty$ is discrete, then we obtain a unique topological group structure on $\widetilde{G}_A$, and $\widetilde{G}_A \to G_A = \widetilde{G}_A/\mathfrak{z}$ is an n-fold covering map because of $K \cap \mathfrak{z} = 1$. In this way we can construct a global covering group $\widetilde{G}_A$ of G_A which coincides locally with the covering stated in Theorem 1. Since $\mathfrak{z}$ can be regarded as a subgroup of $\widetilde{G}_\mathfrak{p}$ for every $\mathfrak{p}$, $\widetilde{G}_A$ is a semi-direct product of $\widetilde{G}_\mathfrak{p}$. The covering $\widetilde{G}_A \to G_A$ is not trivial, because it is not locally trivial at finite places[8]. At infinite places, the covering is trivial, and if $\widetilde{G}_0$ is the inverse image of G_0 with respect to the covering map, then $\widetilde{G}_A = G_\infty \times \widetilde{G}_0$.

Let $\alpha \in G_F$ be a principal adele. Then, $s_{\mathfrak{p}}(\alpha) = 1$ for almost all $\mathfrak{p}$. Therefore $s_A(\alpha) = \prod_{\mathfrak{p}} s_{\mathfrak{p}}(\alpha)$ is well defined. Moreover, $a_{\mathfrak{p}}(\alpha, \beta) = 1$, $(\alpha, \beta \in G_F)$, for almost all $\mathfrak{p}$, and from the product formula of the norm residue symbol follows $\prod_{\mathfrak{p}} a_{\mathfrak{p}}(\alpha, \beta) = 1$. This implies $b_A(\alpha, \beta) = s_A(\alpha)^{-1} s_A(\beta)^{-1} s_A(\alpha\beta)$. So, if we put $\hat{\alpha} = (\alpha, s_A(\alpha))$, $(\alpha \in G_F)$, then

$$\hat{\alpha}\hat{\beta} = (\alpha\beta,\ b_A(\alpha, \beta) s_A(\alpha) s_A(\beta)) = (\alpha\beta,\ s_A(\alpha\beta)) = \widehat{\alpha\beta}$$

for $\alpha, \beta \in G_F$. Thus, $\alpha \to \hat{\alpha}$ gives an isomorphism of G_F onto the group $\hat{G}_F \subset \widetilde{G}_A$ of all $\hat{\alpha}$; $\hat{G}_F$ is a discrete subgroup of $\widetilde{G}_A$.

In the rest of this §, we always assume $n^2 | N$.

Proposition 1. For an element $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $\Gamma = GL(2, \mathfrak{o})_N = G_F \cap K_0 G_\infty$, put

$$\chi(\sigma) = \begin{cases} \left(\frac{c}{d}\right), & \text{if } c \neq 0, \\ 1, & \phantom{\text{if }} c = 0, \end{cases}$$

where $\left(\frac{c}{d}\right)$ is the n-th power residue symbol in F. Then, $s_A(\sigma) = \chi(\sigma)$.

8) See the theorem of [9].

Proof. Suppose first $c \neq 0$. Then, using the definition, the fundamental properties of the power residue symbol[9], and the product formula of the norm residue symbol, we have

$$s_A(\sigma) = \prod_{\mathfrak{p}} s_{\mathfrak{p}}(\sigma) = \prod_{\mathfrak{p}|c} (c, d \det \sigma^{-1})^{-1} = \prod_{\mathfrak{p}\nmid c} (c, d \det \sigma^{-1}) = \left(\frac{c}{d}\right).$$

In case $c = 0$, the proposition is trivial.

Proposition 2[10]. Let χ be as in Prop. 1. Then, $\chi(\sigma\tau) = \chi(\sigma)\chi(\tau)$, $(\sigma, \tau \in \Gamma)$, i.e. χ is a character of Γ.

Proof. Since $(\sigma, 1)$, $(\tau, 1)$ belong to $K_0 G_\infty \subset \tilde{G}_A$, the equality $(\sigma, 1)(\tau, 1) = (\sigma\tau, 1)$ must hold. Hence $b_A(\sigma, \tau) = 1$. On the other hand, $b_A(\sigma, \tau)s_A(\sigma)s_A(\tau) = s_A(\sigma\tau)$ because of $\hat{\sigma}\hat{\tau} = \widehat{\sigma\tau}$. Our assertion follows immediately from this and from Prop. 1.

Proposition 3. Let Γ_1 be the kernel of the character in Prop. 2. Then, $\hat{G}_F \cap K_0 G_\infty = \hat{\Gamma}_1 (= \Gamma_1)$.

Proof. Since $\sigma = (\sigma, \chi(\sigma))$, $(\sigma \in \Gamma)$, by Prop. 1, σ belongs to KG_∞ if and only if and only if $\chi(\sigma) = 1$.

Proposition 4. $\hat{G}_F K_0 G_\infty \mathfrak{z} = \hat{G}_F K_0 G_\infty$.

Proof. Since χ is not trivial, there exists a $\sigma \in \Gamma$ such that

9) See [4].
10) This is the theorem of [7]; it is given here a new proof.

$\chi(\sigma) = \zeta$ is equal to a given n-th root of unity, Γ being as in Prop. 1. It follows now from $(\sigma, \chi(\sigma)) \in \hat{G}_F$, $(\sigma, 1) \in K_0G_\infty$ that $(\sigma, \chi(\sigma))(\sigma, 1)^{-1} = (1, \chi(\sigma)) = \dot{\zeta} \in \hat{G}_FK_0G_\infty$, which proves the proposition.

By virtue of Prop. 4, we can obtain a correspondence between functions f_A on $\hat{G}_F\backslash \hat{G}_FK_0G_\infty\mathfrak{z}/\Delta K$ and the functions f on $\Gamma_1\backslash H^r$, H^r being as in § 1; similarly to (16), it is done by

$$(24) \qquad f(g_\infty) = f_A(g) \qquad \text{for} \qquad g = \hat{\xi}kg_\infty$$

with $\hat{\xi} \in \hat{G}_F$, $k \in K$, $g_\infty \in G_\infty$.

For the compact subgroup $K\mathfrak{z} \subset \tilde{G}_A$, a character χ can be defined by

$$(25) \qquad \chi(k\cdot\dot{\zeta}) = \zeta^{-1}, \qquad (k \in K),$$

and we can consider automorphic functions on $\hat{G}_FK_0G_\infty\mathfrak{z}/\Delta$ with respect to $\hat{G}_F$, $K\mathfrak{z}$, and χ; they are namely functions f_A on $\hat{G}_FK_0G_\infty\mathfrak{z}$ satisfying $f(\gamma x) = f(x)$, $(\gamma \in \hat{G}_F)$, and $f(xk\dot{\zeta}) = f(x)\chi(k\dot{\zeta}) = f(x)\zeta^{-1}$. From now on, we denote the group $\hat{G}_FK_0G_\infty\mathfrak{z}/\Delta = \hat{G}_FKG_\infty/\Delta$ by $\tilde{G}$; $\tilde{G}$ is the inverse image with respect to $\tilde{G}_A \to G_A$ of $G = G_FK_0G_\infty$.

Proposition 5. There is a one-to-one correspondence determined by (24) between automorphic functions f_A on $\tilde{G}$ with respect to $\hat{G}_F$, $K\mathfrak{z}$, and χ, and functions f on H^r with the property

$$(26) \qquad f(\sigma u) = \chi(\sigma)f(u), \qquad (\sigma \in \Gamma),$$

Γ, $\chi(\sigma)$ being as in Prop. 1.

Proof. Let $g = \hat{\xi} k g_\infty$ be as in (24), and put $\chi(\sigma) = \zeta$. Then our assertion follows from $g\dot{\zeta} = \hat{\xi}\dot{\zeta} k g_\infty = \hat{\xi}(\sigma, \chi(\sigma))(\sigma^{-1}, 1) k g_\infty$ $= \widehat{\xi\sigma} \cdot (\sigma^{-1})_0 k \cdot (\sigma^{-1})_\infty g_\infty.$

The formula (26) is an analogy to (15).

§ 4. Hecke ring of the metaplectic group.

Since the metaplectic group $\widetilde{G}_A$ constructed in the preceeding § has the disctete subgroup $\widehat{G}_F$ and the compact subgroup $K\mathfrak{z}$, and since $K\mathfrak{z}$ has the character χ, we can consider the Hecke ring $\text{æ}_\chi(\widetilde{G}_A/\Delta, K\mathfrak{z})$ according to the definition given in § 2 for general topological groups[11]. Also for the group $\widetilde{G} = \widehat{G}_F K_0 G_\infty = \widehat{G}_F K_0 G_\infty \mathfrak{z}$, we can define the Hecke ring $\text{æ}_\chi(\widetilde{G}, K\mathfrak{z})$. In both cases, æ_χ is the tensor product of $\text{æ}(G_\infty, K_\infty)$, with trivial character, and the "finite part" $\text{æ}_{\chi, 0}$. If $\widetilde{G}_0$ stands for the inverse image of G_0 with respect to $\widetilde{G}_A \to G_A$, then the finite part $\text{æ}_{\chi, 0}(\widetilde{G}_A, K\mathfrak{z})$ of $\text{æ}_\chi(\widetilde{G}_A/\Delta, K\mathfrak{z})$ is the same thing as $\text{æ}_\chi(\widetilde{G}_0, K_0\mathfrak{z})$, which is generated over $\mathbf{C}$ by all functions on $\widetilde{G}_0$ satisfying (11) and vanishing outside a set of the form $K(a, 1)K\mathfrak{z}$ with $a \in G_0$. The finite part $\text{æ}_{\chi, 0}(\widetilde{G}, K\mathfrak{z})$ of $\text{æ}_\chi(\widetilde{G}, K\mathfrak{z})$ is the subring of $\text{æ}_\chi(\widetilde{G}_0, K_0\mathfrak{z})$ generated by all functions which vanish outside a set of the form $K(a, 1)K\mathfrak{z}$, where a means an adele in $G_0 \cap G$, $(G = G_F K_0 G_\infty)$. The subring of $\text{æ}_{\chi, 0}$ generated by functions which vanish outside a set of the form $K(a, 1)Kz$ with $a \in G_0$, $pr_0 a = 1$ for all $p|N$, will be denoted by $\text{æ}'_{\chi, 0}$ in the above both cases[12].

The aim of this paper is, as was said in the introduction, to construct a Hilbert space which is generated by certain Eisenstein series, and which gives a finite sum of irreducible unitary representations of the group G_A.

11) Here and already in introducing automorphic functions on $\widetilde{G}$ in § 3, we have avoided such a notation as $K\mathfrak{z}\Delta/\Delta$, and have written simply $K\mathfrak{z}$ for it. Cf, the definition of $\text{æ}_\chi(G, K)$ in § 2.

12) In these definitions, of course, the adele a is not fixed, but may depend freely on functions.

From this point of view, there is at least abstractly no essential difference between the investigation of $\tilde{G}_A$ and that of the subgroup $\tilde{G}$ of finite index. So, we shall mainly treat $\tilde{G}$ in the rest of this paper. In this §, however, we shall state fundamental properties of $æ_\chi$, especially of $æ'_{\chi,0}$ for both $\tilde{G}_A$ and $\tilde{G}$, without strict selection of what is actually used later.

Proposition 6. Let $\mathfrak{p}$ be a finite place of F which does not divide N, N being a multiple of n^2. Then, the factor set $b(\sigma, \tau)$ given by (21) of $G_\mathfrak{p} = GL(2, F_\mathfrak{p})$ has the following properties :

i) if $\sigma = \begin{pmatrix} a & \\ & d \end{pmatrix}$, $\tau = \begin{pmatrix} a' & \\ & d' \end{pmatrix}$ are diagonal elements of $G_\mathfrak{p}$,

then $b(\sigma, \tau) = (a, d')$.

ii) if $\sigma = \begin{pmatrix} a^n & \\ & d^n \end{pmatrix} \in G_\mathfrak{p}$, then $b(\sigma, \tau) = b(\tau, \sigma) = 1$ for every $\tau \in G_\mathfrak{p}$

iii) if the determinants of $\sigma, \tau \in G_\mathfrak{p}$ are both n-th powers in $F_\mathfrak{p}$, and the $\mathfrak{p}$-orders of $x(\sigma)$, $x(\tau)$ and $x(\sigma\tau)$ are divisible by n, then $b(\sigma, \tau) = 1$. Here, $x(\sigma)$ for $\sigma \in G_\mathfrak{p}$ is defined by the same formula as in Theorem 1.

Proof. i) follows directly from the definition[13]. In order to prove ii), it suffices to show $a(\sigma, \tau) = a(\tau, \sigma) = 1$ and $s(\sigma\tau) = s(\tau\sigma) = s(\tau)$. But, these assertions are also direct consequences[13] of the definitions in (19) and in Theorem 2[14]. The proof of iii) is similar.

We now resume the observation of the global metaplectic group $\tilde{G}_A$. In the rest of this §, we always assume that N is divisible by n^2, and

13) Cf. Lemma 1 of [9].
14) See the remark above (21).

we denote by G_0' the group of adeles $a \in G_0$ such that $pr_p a = 1$ for all $p \mid N$.

Proposition 7. If $a \in G_0'$, then the double coset $K\mathfrak{z}(a, 1)K\mathfrak{z}$ in $\widetilde{G}_A$ is a disjoint union of the double cosets of the form $K(a, 1)K \cdot \dot{\zeta}$, $(\dot{\zeta} \in \mathfrak{z})$. The number of distinct cosets $K(a, 1)K \cdot \dot{\zeta}$ in $K\mathfrak{z}(a, 1)K\mathfrak{z}$ is n if and only if the $\mathfrak{p}$-elementary divisors of $a_{\mathfrak{p}}$ are n-th powers in $F_{\mathfrak{p}}$ for all $\mathfrak{p}$.

Proof. The first assertion is obvious. To prove the second assertion, take $\varepsilon_1, \varepsilon_2 \in K_{\mathfrak{p}} \subset G_{\mathfrak{p}}$ such that $\varepsilon_1 a_{\mathfrak{p}} \varepsilon_2 = \begin{pmatrix} e_1 & \\ & e_2 \end{pmatrix}$, where e_1, e_2 are powers of an element $\omega \in \mathfrak{o}_{\mathfrak{p}}$ with $\mathrm{ord}_{\mathfrak{p}} \omega = 1$ such that $e_1 \mid e_2$, and assume first that one of e_i, say e_1, is not an n-th power in $F_{\mathfrak{p}}$. Then, there exists a unit η of $F_{\mathfrak{p}}$ with $(e_1, \eta) = \zeta_1 \neq 1$. From this and from i) of Prop. 6 it follows that

$$\left(\begin{pmatrix} 1 & \\ & \eta \end{pmatrix}, 1\right)(\varepsilon_1 a_{\mathfrak{p}} \varepsilon_2, 1)\dot{\zeta}_1 = (\varepsilon_1 a_{\mathfrak{p}} \varepsilon_2, 1)\left(\begin{pmatrix} 1 & \\ & \eta \end{pmatrix}, 1\right).$$

This means $K(\varepsilon_1 a_{\mathfrak{p}} \varepsilon_2, 1)K \cdot \dot{\zeta}_1 = K(\varepsilon_1 a_{\mathfrak{p}} \varepsilon_2, 1)K$. Since $K(\varepsilon_1 a_{\mathfrak{p}} \varepsilon_2, 1)K$ is one of $K(a, 1)K \cdot \dot{\zeta}$, at least two cosets of the form $K(a, 1)K \cdot \dot{\zeta}$ coincide with each other. Next, assume conversely that e_1, e_2 are n-th powers, and suppose that there exists a $\dot{\zeta}_1 \in \mathfrak{z}$ with $K(a_{\mathfrak{p}}, 1)K = K(a_{\mathfrak{p}}, 1)K \cdot \dot{\zeta}_1$. Then, there are elements $\varepsilon_1', \varepsilon_2' \in K_{\mathfrak{p}} \subset G_{\mathfrak{p}}$ such that

$$(\varepsilon_1', 1)\left(\begin{pmatrix} e_1 & \\ & e_2 \end{pmatrix}, 1\right) = \left(\begin{pmatrix} e_1 & \\ & e_2 \end{pmatrix}, 1\right)(\varepsilon_2', 1)\dot{\zeta}_1.$$

But, this is impossible by ii) of Prop. 6.

Proposition 8. If $a \in G'_0$, and if there exists a place $\mathfrak{p}$ of F such that at least one of the $\mathfrak{p}$-elementary divisors of $a_{\mathfrak{p}}$ is not an n-th power in $F_{\mathfrak{p}}$, then $T((a, \zeta)) = 0$, $(\zeta^n = 1)$.

Proof[15]. *It is no restriction of the generality to assume* $\zeta = 1$. Now, the operator $T((a, 1))$ is determined by a function ψ on $\widetilde{G}_A$ with the property (11) which takes the value 0 outside $K(a, 1)K$. Such a function must be 0 by Prop. 7.

If a is an element of G'_0 such that $a_{\mathfrak{p}}$ is the n-th power of a diagonal element in $G_{\mathfrak{p}}$ for all $\mathfrak{p}$, then we call the double coset $K_0(a, 1)K_0$ a standard double coset. A function ψ_a which takes the value 1 on a sastandard coset $K_0(a, 1)K_0$, $\zeta^{-1} = \chi(\zeta)$ on $K_0(a, 1)K_0 \cdot \zeta$ and 0 everywhere else on $\widetilde{G}_0$ will be called a standard χ-characteristic function. The standard function ψ_a is exactly the function in (11) which determines the Hecke operator $T((a, 1)) \in æ'_{\chi, 0}(G, K\mathfrak{z})$ through $f \circ T((a, 1)) = f * \psi_a$, and the algebra $æ'_{\chi, 0}(G, K\mathfrak{z})$ is a complex linear combination of standard χ-characteristic functions. It should be noted here that the group K in the sense of § 2 is $K\mathfrak{z}$ in the present case, and therefore the measure of $\widetilde{G}_A$ is normalized by $\mu(K) = 1/n$.

Theorem 3. Let $\mathfrak{p}$ be a finite place of F, let (σ, ζ) be an element of $G_{\mathfrak{p}}$, $(\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_{\mathfrak{p}})$, and set

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15) Another proof of this proposition is obtained by using (14). In fact, $\chi(a\tau_j^{-1}a^{-1}\tau_j)$ is a character of the group $K \cap a^{-1}Ka/K_1 \cap a^{-1}K_1a$ which is not trivial in the case treated here. Therefore we have $c_{\chi}^{1}(a) = 0$.

(27) $$(\sigma, \zeta)^* = (\sigma^*, s(\sigma)s(\sigma^*)s'(\sigma)\zeta^{-1}),$$

with $\sigma^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, and

$$s'(\sigma) = \begin{cases} (\det \sigma, -c), & \text{if } c \neq 0, \\ (-1, d \det \sigma), & c = 0. \end{cases}$$

Then, $*$ is an anti-automorphism of $\tilde{G}_{\mathfrak{p}}$ satisfying $(\sigma, \zeta)^{**} = (\sigma, \zeta \cdot (-1, \det \sigma))$, and gives rise to an anti-automorphism of $\tilde{G}_A$. Moreover, if $a \in G_0'$, and every $\mathfrak{p}$-component $a_{\mathfrak{p}}$ is the n-th power of a diagonal element of $G_{\mathfrak{p}}$, then $(K(a,1)K)^* = K(a, 1)K$ in $\tilde{G}_A$.

Proof. We use temporarily the notation $(\sigma, \zeta)_a$ in the proof of Theorem 1, so that (27) can be reduced to $(\sigma, \zeta)_a^* = (\sigma^*, \zeta^{-1}s'(\sigma))_a$, and we show first that $*$ defines an anti-automorphism of $SL(2, F_{\mathfrak{p}})$. Let $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\tau = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ be in $SL(2, F_{\mathfrak{p}})$, and put $\sigma\tau = \begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix}$. If $cc'c'' \neq 0$, then $a(\sigma, \tau)a(\tau^*, \sigma^*) = (c, c')(-c^{-1}c', c'')(-c', -c)\cdot(-c'^{-1}c, -c'') = 1$, if $c = 0$, $c'c'' \neq 0$, then $a(\sigma, \tau)a(\tau^*, \sigma^*) = (c', d)(-c', d^{-1}) = (-1, d)$, if $c' = 0$, $cc'' \neq 0$, then $a(\sigma, \tau)a(\tau^*, \sigma^*) = (c, d')(-c, d'^{-1}) = (-1, d')$, and if $cc' \neq 0$, $c'' = 0$, then $\tau = \sigma^{-1}(\sigma\tau) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a'' & b'' \\ & d'' \end{pmatrix}$ and we have $a(\sigma, \tau)a(\tau^*, \sigma^*) = (c, -cd''^{-1})(d''^{-1}, d'')(cd''^{-1}, -c)(d'', d''^{-1}) = (-1, d'')$. Furthermore, if $c = c' = c'' = 0$, then $a(\sigma, \tau)a(\tau^*, \sigma^*) = 1$[16]. Since

16) For these computations, use Lemma 1 of [9].

$a(\sigma, \tau) = a(\tau^*, \sigma^*)^{-1} s'(\sigma)^{-1} s'(\tau)^{-1} s'(\sigma\tau)$ holds in all cases,

$(\sigma, \zeta)_a \to (\sigma, \zeta)_a^* = (\sigma^*, \zeta^{-1} s'(\sigma))_a$ is an anti-automorphism of $SL(2, F_{\mathfrak{p}})$.

We now intend to prove that $*$ in (27) is an anti-automorphism of $\tilde{G}_{\mathfrak{p}}$ satisfying $(\begin{pmatrix} 1 & \\ & y \end{pmatrix}, 1)_a^* = (\begin{pmatrix} y & \\ & 1 \end{pmatrix}, 1)_a$.

In fact, the relation

$$(\begin{pmatrix} 1 & \\ & y \end{pmatrix}\sigma, 1)_a^* = \{(\begin{pmatrix} 1 & \\ & y \end{pmatrix}, 1)_a(\sigma, 1)_a\}^* = (\sigma^*, s'(\sigma))_a(\begin{pmatrix} y & \\ & 1 \end{pmatrix}, 1)_a$$

holds for any $\sigma \in SL(2, F_{\mathfrak{p}})$ because of

$$a(\sigma^*, \begin{pmatrix} y & \\ & 1 \end{pmatrix})_a = \begin{cases} (c, y)^{-1}, & c \neq 0, \\ 1, & c = 0. \end{cases}$$

Therefore, to prove that $*$ is an anti-automorphism of $\tilde{G}_{\mathfrak{p}}$ it is enough to show that $*$ satisfies

$$(28) \qquad \{(\sigma, 1)_a(\begin{pmatrix} 1 & \\ & y \end{pmatrix}, 1)_a\}^* = (\begin{pmatrix} y & \\ & 1 \end{pmatrix}, 1)_a(\sigma^*, s'(\sigma))_a$$

for any $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, F_{\mathfrak{p}})$. If $c \neq 0$, then the left hand side of (28) is

$$(\sigma\begin{pmatrix} 1 & \\ & y \end{pmatrix}, 1)_a^* = (\begin{pmatrix} y & \\ & 1 \end{pmatrix}\sigma^*, (y, -c))_a,$$

and $(y, -c)$ is equal to $s'(\sigma)a(\begin{pmatrix} y & \\ & 1 \end{pmatrix}, \sigma^*)$. If $c = 0$, then the left hand side of (28) is

$$(\sigma(\begin{smallmatrix}1 & \\ & y\end{smallmatrix}),\ (y,\ d))^*_a = ((\begin{smallmatrix}y & \\ & 1\end{smallmatrix})\sigma^*,\ (-1,\ d)(y,\ d)^{-1})_a,$$

and $(-1,\ d)(y,\ d^{-1}) = (d,\ -y)$ is equal to $s'(\sigma)a((\begin{smallmatrix}y & \\ & 1\end{smallmatrix}),\ \sigma^*)$ in this case. Thus (28) holds always. Since, $s(\sigma)^{-1}s(\sigma^*) = (-1,\ \det\sigma)$ follows directly from the definition of $s(\sigma)$ for $\sigma \in G_{\mathfrak{p}}$, the first assertion of the theorem is proved.

To prove the second assertion of the theorem, it is enough to show that $K^* = K$, because we have $K(a,\ 1)^*K = K(a^*,\ 1)K = K(a,\ 1)K$, whenever $a_{\mathfrak{p}}$ is the n-th power of a diagonal element of $G_{\mathfrak{p}}$ for all $\mathfrak{p}$. We observe an element $(\sigma,\ 1)$ of $K_{\mathfrak{p}} \subset \widetilde{G}_{\mathfrak{p}}$ for a finite place $\mathfrak{p}$, where $\sigma = (\begin{smallmatrix}a & b\\ c & d\end{smallmatrix})$ is an element of $K_{\mathfrak{p}} \subset G_{\mathfrak{p}}$. Since $s(\sigma\sigma^*) = 1$, it follows from Theorem 2 that $s(\sigma)s(\sigma^*)s'(\sigma) = a(\sigma,\ \sigma^*)s'(\sigma)$. Furthermore, if $c \neq 0$, then $a(\sigma,\ \sigma^*) = (c \det\sigma^{-2},\ -c\det\sigma^{-1})(\det\sigma,\ \det\sigma^{-1}) = (-c,\ \det\sigma)$, and if $c = 0$, then $a(\sigma,\ \sigma^*) = (d\det\sigma^{-1},\ d^{-1})^{-1}(\det\sigma,\ d\det\sigma^{-1}) = (-1,\ d\det\sigma)$. Hence $a(\sigma,\ \sigma^*)s'(\sigma) = 1$, and consequently we have $(\sigma,\ 1)^* = (\sigma^*,\ 1)$. This completes the proof of the theorem.

In the beginning of this §, we have defined the algebra $\mathfrak{a}'_{\chi,0}(\widetilde{G}_A,\ K\mathfrak{z})$. The next theorem about this algebra may be proved in defferent ways. We give here, however, a direct proof using Theorem 3.

Theorem 4. The algebra $\mathfrak{a}'_{\chi,0}(\widetilde{G}_A,\ K\mathfrak{z})$ is commutative.

Proof. Let a, b be two elements of G_0' such that, for every $\mathfrak{p}$, $a_\mathfrak{p}$, $b_\mathfrak{p}$ are of the form $\begin{pmatrix} e_1^n & \\ & e_2^n \end{pmatrix}$, $\begin{pmatrix} f_1^n & \\ & f_2^n \end{pmatrix}$, respectively, with e_i, $f_i \in F_\mathfrak{p}$, and let ψ_a, ψ_b be standard χ-characteristic functions with $\psi_a((a, 1)) = \psi_b((b, 1)) = 1$ in the sense explained after Prop. 8. To prove our theorem, it is enough to show $\psi_a * \psi_b = \psi_b * \psi_a$. First, we shall see that $\psi_a * \psi_b$ is a linear combination of standard χ-characteristic functions with positive, rational integral coefficients. For this purpose, we observe the coset decomposition $K_0\mathfrak{z}(b, 1)K_0\mathfrak{z} = \cup K_0\mathfrak{z}(b, 1)\sigma_i$ with $\sigma_i \in K_0 \subset \tilde{G}_\mathfrak{p}$. If we regard σ_i as an element of G_A, then the set $\{\sigma_i\}$ is a set of representatives of $b^{-1}K_0b \cap K_0 \backslash K_0$ in G_A, and it is no restriction of the generality to assume that $f_1 | f_2$ for every $\mathfrak{p}$. So, we may choose σ_i such that every local component of σ_i is either of the form $\begin{pmatrix} 1 & z \\ & 1 \end{pmatrix}$ or $\begin{pmatrix} z & -1 \\ 1 & \end{pmatrix}$. Now, let x be an element of G_0' such that every $\mathfrak{p}$-component $x_\mathfrak{p}$ is of the form $\begin{pmatrix} x_1^n & \\ & x_2^n \end{pmatrix}$ with $x_i \in F_\mathfrak{p}$, and assume $x_2 | x_1$. Then,

$$\psi_a * \psi_b((x, 1)) = \int_{\tilde{G}_A} \psi_a((x, 1)y^{-1})\psi_b(y)dy = \sum \psi_a((x, 1)\,\sigma_i^{-1}(b, 1)^{-1})$$

and $\psi_a((x, 1)\sigma_i^{-1}(b, 1)^{-1}) = \psi_a((x\sigma_i^{-1}b^{-1}, 1))$ by ii) of Prop. 6. For every $\mathfrak{p}$, *the* $\mathfrak{p}$-*component* σ *of* $x\sigma_i^{-1}b^{-1}$ *is one of the following two forms :*

$$\text{i)} \quad \sigma = \begin{pmatrix} x_1^n f_1^{-n} & x_1^n f_2^{-n} z \\ & x_2^n f_2^{-n} \end{pmatrix}, \qquad \text{ii)} \quad \sigma = \begin{pmatrix} & x_1^n f_2^{-n} \\ -x_2^n f_1^{-n} & x_2^n f_2^{-n} z \end{pmatrix}.$$

In the case i), $\tau\sigma$ is diagonal with $\tau = \begin{pmatrix} 1 & -x_1^n x_2^{-n} z \\ & 1 \end{pmatrix} \in K_{\mathfrak{p}}$, and from the definition we can deduce $b(\tau, \sigma) = 1$[17]. In the case ii), the situation is somewhat different. If $\mathrm{ord}_{\mathfrak{p}} f_1^{-n} \leqq \mathrm{ord}_{\mathfrak{p}} f_2^{-n} z$, then $\sigma\tau$ is diagonal with $\tau = \begin{pmatrix} 1 & f_1^n f_2^{-n} z \\ & 1 \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \in K_{\mathfrak{p}}$, and $b(\sigma, \tau) = 1$. If $\mathrm{ord}_{\mathfrak{p}} x_1^n \leqq \mathrm{ord}_{\mathfrak{p}} x_2^n z$, then $\tau\sigma$ is diagonal with $\tau = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \begin{pmatrix} 1 & \\ -x_1^{-n} x_2^n z & 1 \end{pmatrix}$, and $b(\tau, \sigma) = 1$.

If finally $\mathrm{ord}_{\mathfrak{p}} x_2^n f_2^{-n} z$ is smaller than the $\mathfrak{p}$-order of any other matric element in the case ii), then we may assume that $\mathrm{ord}_{\mathfrak{p}} z$ is divisible by n; otherwise the elementary divisors of σ are not n-th powers, and therefore $\psi_a(\sigma) = 0$ by Prop. 8. If $n \mid \mathrm{ord}_{\mathfrak{p}} z$, then $\tau_1 \sigma \tau_2$ is diagonal with

$\tau_1 = \begin{pmatrix} 1 & -x_1^n x_2^{-n} z^{-1} \\ & 1 \end{pmatrix} \in K_{\mathfrak{p}}$, $\tau_2 = \begin{pmatrix} 1 & \\ f_1^{-n} f_2^n z^{-1} & 1 \end{pmatrix} \in K_{\mathfrak{p}}$, and

$b(\tau_1, \sigma) = b(\tau_1\sigma, \tau_2) = 1$. Thus, both in case i) and in case ii), it was veirified that $(x\sigma_i^{-1} b^{-1}, 1)$ belongs to a standard double coset, i. e. a coset of the form $K_0(w^n, 1)K_0 \subset \tilde{G}_0$, every $w_{\mathfrak{p}}$ being diagonal. Since ψ_a is a standard χ-characteristic function, $\psi_a(\sigma)$ is either 1 or 0. This implies that $\psi_a * \psi_b$ is a linear combiantion of standard χ-characteristic functions with positive, ratioanl integral coefficients as claimed above. Consider now the mapping of $æ'_{\chi, 0}(\tilde{G}_A, K_{\mathfrak{F}})$ into itself defined by $\psi(g) \to \psi^*(g) = \overline{\psi(g^*)}$, $(g \in \tilde{G}_0)$. This is a linear mapping over $\mathbf{R}$, and an anti-automorphism of the algebra; namely, $(\psi_1 * \psi_2)^* = \psi_2^* * \psi_1^*$ holds for $\psi_1, \psi_2 \in æ'_{\chi, 0}$. However, Theorem 3 implies $\psi^* = \psi$ for any standard χ-characteristic function ψ.

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17) Here and in the following calculations of $b(\sigma, \tau)$, use iii) of Prop. 6.

Therefore $\psi_a^* = \psi_a$, $\psi_b^* = \psi_b$, and from the above argument about $\psi_a * \psi_a$ it also follows that $(\psi_a * \psi_b)^* = \psi_a * \psi_b$. Hence we have $\psi_a * \psi_b = \psi_a * \psi_b$, and the theorem is proved.

We conclude this § by giving a concrete formula similar to (17) about the action of a special Hecke operator. Let ω be a prime number of F such that $\omega \equiv 1 \pmod N$, and let a be the adele of which the ω-component is $\begin{pmatrix} 1 & \\ & \omega^{nt} \end{pmatrix}$, $(0 \leqq t)$, and all other components are identity. The operator $T((a, 1))$[18] will then be denoted by $T(1, \omega^{nt})$. We use formula (14) with $K_{\mathfrak{z}}$, K instead of K, K_1. Since $(a, 1)^{-1}K(a, 1) = (a^{-1}Ka, 1)$ by ii) of Prop. 6, we have $K_{\mathfrak{z}} \cap (a, 1)^{-1}K_{\mathfrak{z}}(a, 1) = \bigcup_{\zeta^n = 1} (K \cap (a, 1)^{-1}K(a, 1) \cdot \dot{\zeta}$. So, $c_\chi((a, 1)) = (K : K \cap a^{-1}Ka)^{-1}$, *the index being taken in* G_A. *On the other hand*, if $K = \cup (K \cap a^{-1}Ka)\sigma_i$ is a coset decomposition in G_A, then $K_{\mathfrak{z}} = \bigcup (K_{\mathfrak{z}} \cap (a, 1)^{-1}K_{\mathfrak{z}}(a, 1))(\sigma_i, 1)$ is a coset decomposition. Furthermore, $q = ||\omega||$ being the norm of ω, we can choose as $\{\sigma_i\}$ the $q^{nt} + q^{nt-1}$ adeles whose ω-components are $\begin{pmatrix} 1 & z \\ & 1 \end{pmatrix}$, $(\mathfrak{o}_\omega \ni z \bmod \omega^{nt})$, $\begin{pmatrix} z & -1 \\ 1 & \end{pmatrix}$, $(\omega | z \bmod \omega^{nt})$, and whose other components are identity. Suppose $z = \omega^k c$, $k = \operatorname{ord}_\omega z$, in the second case, and put $\sigma = \begin{pmatrix} z & -1 \\ \omega^{nt} & \end{pmatrix}$, then we have

$$\tau\sigma = \begin{pmatrix} \omega^k & -c^{-1} \\ & \omega^{nt-k} \end{pmatrix} \quad \text{with} \quad \tau = \begin{pmatrix} c^{-1} & \\ -\omega^{nt-k} & c \end{pmatrix},$$

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18) See foot note 3.

and $b(\tau, \sigma) = (c, \omega)^k$. This means

$$K(\begin{pmatrix} 1 & \\ & \omega^{nt} \end{pmatrix}, 1)(\begin{pmatrix} z & -1 \\ 1 & \end{pmatrix}, 1) = K(\begin{pmatrix} \omega^k & -c^{-1} \\ & \omega^{nt-k} \end{pmatrix}, (c, \omega)^k).$$

Therefore, for a function f_A on $\tilde{G}_A$ with (25), (14) yields

$$(29) \qquad (q^{nt} + q^{nt-1}) \cdot f_A(g) \circ T(1, \omega^{nt})$$
$$= f_A(g(a_{0,0}, 1)^{-1}) + \sum_{k=1}^{nt-1} \sum_{\substack{\mathfrak{o} \ni c \bmod \omega^k \\ \omega \nmid c}} f_A(g(a_{k,c}, 1)^{-1})(\frac{c}{\omega})^k$$
$$+ \sum_{\mathfrak{o} \ni c \bmod \omega^{nt}} f_A(g(a_{nt,c}, 1)^{-1}),$$

where $a_{k, c}$ stands for the adele whose only one non-vanishing component is the ω-component $\begin{pmatrix} \omega^{nt-k} & c \\ & \omega^k \end{pmatrix}$. If we take an automorphic function f_A on $\hat{G}_F \backslash \tilde{G}/\Delta K$, and observe the influence of $T(1, \omega^{nt})$ on the function $f(u)$ on $\Gamma \backslash H^r$ induced by f_A through the relation (24), then (29) yields

$$(30) \qquad (q^{nt} + q^{nt-1}) \cdot f(u) \circ T(1, \omega^{nt}) = f(\begin{pmatrix} \omega^{nt} & \\ & 1 \end{pmatrix} u) +$$
$$\sum_{k=1}^{nt-1} \sum_{\substack{\mathfrak{o} \ni c \bmod \omega^k \\ \omega \nmid c}} f(\begin{pmatrix} \omega^{nt-k} & c \\ & \omega^k \end{pmatrix} u)(\frac{c}{\omega})^k + \sum_{\mathfrak{o} \ni c \bmod \omega^{nt}} f(\begin{pmatrix} 1 & c \\ & \omega^{nt} \end{pmatrix} u).$$

This formula is an analogy of (17); it can be interpreted by means of a double coset with respect to a discontinuous group actiong on H^r, as was done in § 2 about the formula (17). For this purpose, however, one should consider in the case of (30) double cosets with respect to Γ_1, the kernel of χ in Prop. 1, instead of double cosets with respect to $\Gamma = SL(2, \mathfrak{o})_N$.

Using the matrix $\begin{pmatrix} \omega^{n\ell} & \\ & \omega^{n(\ell+t)} \end{pmatrix}$ instead of $\begin{pmatrix} 1 & \\ & \omega^{nt} \end{pmatrix}$ used obove we can define the operator $T(\omega^{n\ell}, \omega^{n(\ell+t)})$. But, this is equal to $T(1, \omega^{nt})$, because if a' is the adele of which the ω-component is $\begin{pmatrix} \omega^{n\ell} & \\ & \omega^{n\ell} \end{pmatrix}$, and other components are 1, then a' belongs to the center of $\widetilde{G}_A$, and every function f_A on $\widehat{G}_F \backslash \widetilde{G}_A / \Delta K$ has the property $f_A(ga') = f_A(g)$.

§ 5. Eisenstein series.

We now observe the product H^r of r copies of the upper-half space, and the group $\Gamma = GL(2, \mathfrak{o})_N$ which acts discontinuously on H^r. As in § 1, $\mathfrak{o}$ stands for the ring of integers of a totally imaginary number field F containing the n-th roots of unity, and Γ is the congruence subgroup mod. N of $GL(2, \mathfrak{o})$; the action on H^r of $\sigma \in \Gamma$ is, similarly to Hilbert's modular group, given by (5). So far as no contrary is stated, we assume always that N is a natural number divisible by n^2. Our aim in this § is to define Eisenstein series related to Γ and the character χ of Γ given in Prop. 2, and to prepare several fundamental formulas for such Eisenstein series.

According to the expression adopted in § 1, the i-th component u_i of $u = (u_1, \ldots, u_r) \in H^r$ will be denoted by $u_i = \begin{pmatrix} z_i & -v_i \\ v_i & \bar{z}_i \end{pmatrix}$ with $z_i \in \mathbf{C}$, $\mathbf{R} \ni v_i > 0$. We shall, however, sometimes use u_i with $v_i \leqq 0$ for the sake of convenience, and identify a $\gamma \in F$ with the point $\gamma = (\gamma^{(1)}, \ldots, \gamma^{(r)}) \in \mathbf{C} \times \ldots \times \mathbf{C} \subset \mathbf{R}^3 \times \ldots \times \mathbf{R}^3$, where $\gamma^{(i)}$ is the i-th conjugate of γ, $\mathbf{R}^3$ means the space of all $\begin{pmatrix} z & -v \\ v & \bar{z} \end{pmatrix}$ with $z \in \mathbf{C}$, $v \in \mathbf{R}$, and $\mathbf{C}$ is imbedded in $\mathbf{R}^3$ by the mapping $w \to \tilde{w} = \begin{pmatrix} w & \\ & \bar{w} \end{pmatrix}$ for $w \in \mathbf{C}$.

For $u = (u_1, \ldots, u_r) \in H^r$, we put $v(u) = v_1 \ldots v_r$, and

$$(31) \qquad j(\sigma, u) = \Pi j_i(\sigma, u), \qquad j_i(\sigma, u) = |c^{(i)} z_i + d^{(i)}|^2 + |c^{(i)}|^2 v_i^2,$$

with $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, F)$. The formula (3) of [10] yields then

$$(32) \qquad v(\sigma u) = j(\sigma, u)^{-1} v(u)$$

for $\sigma \in GL(2, F)$ such that $\varepsilon = \det \sigma$ is a unit of F. Since (3) of [10] is valid only for $\sigma \in SL(2, \mathbf{C})$, (32) is not quite evident; to prove it, one should recall that the action of $\sigma \in GL(2, \mathbf{C})$ on H is given by (3) with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (\det \sigma)^{-1/2}\sigma$, and that $\prod_i |\varepsilon^{(i)}| = 1$ holds for a unit ε of F. Since $j(\sigma, u)$ does not depend on a and b, we shall also write $j(c, d; u)$ for $j(\sigma, u)$. If B is the group of all $\begin{pmatrix} \varepsilon & b \\ & \varepsilon' \end{pmatrix} \in GL(2, F)$ where $\varepsilon, \varepsilon'$ are units of F, then it follows from (32) that

$$(33) \qquad v(\sigma u) = v(u), \qquad (\sigma \in B).$$

Let $\varkappa$ be a cusp of Γ, and $\Gamma_\varkappa$ be the group of all $\sigma \in \Gamma$ with $\sigma\varkappa = \varkappa$. On the other hand, let ρ be an element of $SL(2, F)$ such that $\rho\infty = \varkappa$. Then, $\rho^{-1}\Gamma_\varkappa\rho$ consists of all elements in $\rho^{-1}\Gamma\rho$ which are of the form $\begin{pmatrix} a & b \\ & d \end{pmatrix}$. Since, however, the characteristic roots of $\sigma \in \Gamma$ are algebraic integers, one sees $\rho^{-1}\Gamma_\varkappa\rho = B \cap \rho^{-1}\Gamma\rho$, or $\Gamma_\varkappa = \rho B \rho^{-1} \cap \Gamma$. We have moreover the following

Proposition 9. Notations being as defined above, the group $\rho^{-1}\Gamma_\varkappa\rho = B \cap \rho^{-1}\Gamma\rho$ contains a subgroup Γ_0 of finite index consisting of all elements of the form $\begin{pmatrix} \varepsilon_1 & \mu \\ & \varepsilon_2 \end{pmatrix}$, where $\varepsilon_1, \varepsilon_2$ are units of F satisfying $\varepsilon_1 \equiv \varepsilon_2 \equiv 1 \pmod{N'}$ with a natural number N', and μ belongs to an integral or fractional ideal $\mathfrak{m}$ of F.

Proof. Put $\rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then, $\rho^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, and

$$\rho^{-1}\begin{pmatrix}\varepsilon_1 & \mu \\ & \varepsilon_2\end{pmatrix}\rho = \begin{pmatrix} ad\varepsilon_1 - bc\varepsilon_2 - ac\mu, & ab(\varepsilon_2 - \varepsilon_1) + a^2\mu \\ cd(\varepsilon_1 - \varepsilon_2) - c^2\mu, & ad\varepsilon_2 - bc\varepsilon_1 + ac\mu \end{pmatrix}.$$

From this follows at once the assertion.

We observe again a cusp $\varkappa$ of Γ, and put as above $\Gamma_\varkappa = \{\sigma \in \Gamma \mid \sigma\varkappa = \varkappa\}$. If the character χ in Prop. 2 is trivial on Γ , then $\varkappa$ is said to be an essential cusp (with respect to χ). Let $\varkappa_1, \varkappa_2, \dots, \varkappa_h$ be representatives of all Γ-inequivalent essential cusps, put $\Gamma_{\varkappa_i} = \Gamma_i$, and denote by σ_i an element of $SL(2, F)$ such that $\sigma_i\infty = \varkappa_i$[19]. If here $\varkappa_i = \infty$, let us take $\sigma_i = 1$. Then, the series

$$(34) \qquad E_i(u, s, \chi) = \sum_{\sigma \in \Gamma_i \backslash \Gamma} \overline{\chi}(\sigma) v(\sigma_i^{-1}\sigma u)^s, \qquad (s \in \mathbf{C}),$$

is well-defined by (32), and by the definition of an essential cusp; it is convergent for $\operatorname{Re} s > 2$, and is an eigenfunction of all Laplacians of the space H^r. Furthermore, E_i is automorphic, i. e.

$$(35) \qquad E_i(\sigma u, s, \chi) = \chi(\sigma) E_i(u, s, \chi), \qquad (\sigma \in \Gamma).$$

We call E_i the Eisenstein series attached to the essential cusp $\varkappa_i$. The following theorem gives a more concrete feature of the series E_i.

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19) σ_i corresponds to ρ in Prop. 9.

Theorem 5. Denote by $\mathfrak{e}_{F, N'}$ the group of units of F congruent to 1 modulo a natural number N', let $\mathfrak{C}$ be a set of representatives of $F^{\times}/\mathfrak{e}_{F, N'}$, $F^{\times}$ being the group of non-zero numbers of F, and let M_{ij} be the set of pairs (c, d) of numbers of F such that $c \in \mathfrak{C}$ and that there exists an element in $\sigma_i^{-1}\Gamma\sigma_j$ which is of the form $\begin{pmatrix} * & * \\ c & d \end{pmatrix}$. Then, for a suitable N', there exists a natural number k_i for each essential cusp $\varkappa_i$ of Γ such that

$$(36) \qquad E_i(\sigma_j u, s, \chi) = \delta_{ij} v(u)^s + k_i^{-1} \sum_{(c, d) \in M_{ij}} \overline{\chi}_{ij}(c, d) \frac{v(u)^s}{j(c, d; u)^s},$$

where $\chi_{ij}(c, d) = \chi(\sigma_i \sigma \sigma_j^{-1})$ for $\sigma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \sigma_i^{-1}\Gamma\sigma_j$, and $\delta_{ij} = 1$, or 0 according to $i = j$ or not.

Proof. Let $\mathfrak{e}_i$ be the group of units ε_2 in F with $\begin{pmatrix} \varepsilon_1 & \nu \\ & \varepsilon_2 \end{pmatrix} \in \sigma_i^{-1}\Gamma_i\sigma_i$, and choose N' such that $\mathfrak{e}_{F, N'}$ is contained in all $\mathfrak{e}_i$; this is possible by Prop. 9. Then, for two elements $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\sigma' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \sigma_i^{-1}\Gamma\sigma_j$, the relation $\sigma_i^{-1}\Gamma_i\sigma_i \cdot \sigma = \sigma_i^{-1}\Gamma_i\sigma_i \cdot \sigma'$ holds if and only if there exists an $\varepsilon \in \mathfrak{e}_i$ with $c = \varepsilon c'$, $d = \varepsilon d'$. In other words, the equivalence classes defined by the equivalence relation : $(c, d) \sim (\varepsilon c, \varepsilon d)$, $(\varepsilon \in \mathfrak{e}_i)$, of the pairs of numbers (c, d), $(c, d \in F)$, with $\begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \sigma_i^{-1}\Gamma\sigma_j$ are in one-to-one correspondence with the set of cosets of $\sigma_i^{-1}\Gamma_i\sigma_i \backslash \sigma_i^{-1}\Gamma\sigma_j$. Moreover, the pair $(0, d)$ occurs if and only if $i = j$ because the cusps $\varkappa_i$ are mutually Γ-inequivalent. Thus, our theorem follows from

$$E_i(\sigma_j u, s, \chi) = \sum_{\sigma \in \sigma_i^{-1}\Gamma_i\sigma_i \backslash \sigma_i^{-1}\Gamma\sigma_j} \overline{\chi}(\sigma_i\sigma\sigma_j^{-1}) \frac{v(u)^s}{j(c, d; u)^s} .$$

if we put $k_i = (\mathfrak{e}_i : \mathfrak{e}_{F, N})$. The fact that $\chi(\sigma_i\sigma\sigma_j^{-1})$ depends only on c, d is an immediate consequence of that $\varkappa_i$ is an essential cusp.

In this proof, we have chosen N' such that Prop. 9 holds simultaneously for all essential cusps. We now choose also one and the same ideal $\mathfrak{m}$ of F so that Prop. 9 can hold for all $\varkappa_i$. Since the function $E_i(\sigma_j u, s, \chi)$ is invariant under the transformation $u \to \begin{pmatrix} 1 & \mu \\ & 1 \end{pmatrix} u$, $(\mu \in \mathfrak{m})$, by Prop. 9, we can consider a Fourier expansion of $E_i(\sigma_j u, s, \chi)$. Namely, let $\mathfrak{m}^*$ be the dual ideal of $\mathfrak{m}$ consisting of all numbers $b \in F$ such that $tr_F\, b\mu \in \mathbf{Z}$ for all $\mu \in \mathfrak{m}$, tr_F being the ordinary trace from F to $\mathbf{Q}$. On the other hand, put

$$e(u) = \exp(2\pi\sqrt{-1} \sum_{i=1}^{r} tr\, u_i), \qquad \text{for} \qquad u = (u_1, \ldots, u_r) \in H^r,$$

and

$$(37) \qquad e(m, u) = \exp(2\pi\sqrt{-1} \sum_{i=1}^{r} tr \begin{pmatrix} m^{(i)} z_i & -|m^{(i)}| v_i \\ |m^{(i)}| v_i & \overline{m}^{(i)} \overline{z}_i \end{pmatrix}),$$

for $m \in F$, tr being the trace of a matrix. Then, if $m \neq 0$, (37) is equal to $e(\begin{pmatrix} m & \\ & 1 \end{pmatrix} u)$, the additivity $e(m + m', u) = e(m, u) + e(m', u)$ holds for any $m, m' \in F$, and we have the Fourier expansion

$$(38) \qquad E_i(\sigma_j u, s, \chi) = \sum_{m \in \mathfrak{m}^*} a_{ij, m}(v_1, \dots, v_r, s, \chi) e(m, u)$$

with

$$a_{ij, m}(v_1, \dots, v_r, s, \chi) = V(\mathfrak{m})^{-1} \int_P E_i(\sigma_j u, s, \chi) e(-m, u) dV(z),$$

where $m \in \mathfrak{m}$ is regarded to operate naturally on $\mathbf{C}^r$ through the operation of $\begin{pmatrix} 1 & m \\ & 1 \end{pmatrix}$ on $\mathbf{R}^{3r}$, and $V(\mathfrak{m})$ is the volume with respect to the Euclidean measure dV of a fundamental domain $P = \mathfrak{m} \backslash \mathbf{C}^r$ of the operation. We now propose to give an explicit expression for the Fourier coefficient $a_{ij, m}$. We observe first the case of $m = 0$, i. e. the constant term of the Fourier series. By using (36) in Theorem 5, it is shown that

$$V(\mathfrak{m}) a_{ij, 0}(v_1, \dots, v_r, s, \chi) = \int_P E_i(\sigma_j u, s, \chi) dV(z)$$

$$= V(\mathfrak{m}) \delta_{ij} v(u)^s +$$

$$k_i^{-1} \sum_{c \in \mathfrak{C}} \frac{1}{\| c \|^s} \Big(\sum_{\substack{(c, d) \in M_{ij} \\ d \bmod c\mathfrak{m}}} \bar{\chi}_{ij}(c, d) \Big) \cdot \int_{\mathbf{C}^n} \prod_{i=1}^{n} \frac{v_i^s}{(|z_i|^2 + v_i^2)^s} dV(z),$$

$\| c \|$ being the ordinary norm of c with respect to $F/\mathbf{Q}$. Here, we have used the fact that the pair $(c, d + cm)$, $(m \in \mathfrak{m})$, belongs to M_{ij} whenever the pair (c, d) does, and that $\chi_{ij}(c, d)$ depends only on $d \bmod c$; these properties are consequences of Prop. 9. From the above equalities, it follows that

$$(39) \qquad a_{ij,\,0}(v_1, \ldots, v_r, s, \chi) = \delta_{ij} v(u)^s + \varphi_{ij}(s, \chi) v(u)^{2-s}$$

with

$$(40) \qquad \varphi_{ij}(s, \chi) = V(m)^{-1} k_i^{-1} \sum_{c \in \mathfrak{C}} \frac{1}{\|c\|^s} \Big(\sum_{\substack{(c,\, d) \in M_{ij} \\ d \bmod c\mathfrak{m}}} \overline{\chi}_{ij}(c, d) \Big) \cdot \frac{\pi^r}{(s-1)^r}$$

containing a Dirichlet series. The non-constant terms $a_{ij,\,m}(v_1, \ldots, v_r, s, \chi)$, $(m \neq 0)$, can be treated similarly. We have first

$$V(m) a_{ij,\,m}(v_1, \ldots, v_r, s, \chi)$$

$$= k_i^{-1} \sum_{c \in \mathfrak{C}} \frac{1}{\|c\|^s} \Big(\sum_{\substack{(c,\, d) \in M_{ij} \\ d \bmod c\mathfrak{m}}} \overline{\chi}_{ij}(c, d) \exp(2\pi \sqrt{-1}\, \mathrm{tr}_F \frac{md}{c}) \Big)$$

$$\cdot \int_{\mathbf{C}^r} \prod_{i=1}^{r} \frac{v_i^s}{(|z_i|^2 + v_i^2)^s} \exp(-2\pi\sqrt{-1}(m^{(i)} z_i + \bar{m}^{(i)} \bar{z}_i)) dV(z),$$

and using the formula

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\exp(-2\pi\sqrt{-1}(wz + \bar{w}\bar{z}))}{(x^2 + y^2 + 1)^s} dxdy =$$

$$(2\pi)^s |w|^{s-1} \Gamma(s)^{-1} K_{s-1}(4\pi|w|), \qquad (z = x + \sqrt{-1}\, y,\ w \in \mathbf{C},\ w \neq 0),$$

concerning the modified Bessel function K_s, we obtain

$$(41) \qquad a_{ij,m}(v_1, \ldots, v_r, s, \chi) =$$

$$\varphi_{ij}(s, m, \chi)\|m\|^{\frac{s-1}{2}} \cdot \prod_{i=1}^{r} K_{s-1}(4\pi|m^{(i)}|v_i)\cdot(2\pi)^{rs}\Gamma(s)^{-r}v(u),$$

with the Dirichlet series

$$(42) \qquad \varphi_{ij}(s, m, \chi) =$$

$$V(m)^{-1}k_i^{-1}\sum_{c \in \mathfrak{C}} \frac{1}{\|c\|^s}\Big(\sum_{\substack{(c,d)\in M_{ij} \\ d \bmod c\mathfrak{m}}} \overline{\chi}_{ij}(c, d)\exp(2\pi\sqrt{-1}\,\mathrm{tr}_F\frac{md}{c})\Big).$$

The series (38) is called the Fourier expansion of E_i at the essential cusp $\varkappa_j$. To get a nice survey of the constant terms for this expansion, it is convenient to use the matrix.

$$(43) \qquad Iv(u)^s + \Phi(s, \chi)v(u)^{2-s},$$

where I is the identity matrix of degree h, and $\Phi(s, \chi) = (\varphi_{ij}(s, \chi))$; the element in the i-th line and in the j-th column of (43) is equal to the constant term of the Fourier series for E_i at $\varkappa_j$.

Denote by $\varepsilon(u, s, \chi)$ the column vector ${}^t(E_1, \ldots, E_h)$, then the general theory of the Eisenstein series yields the functional equations

$$\Phi(s, \chi)\Phi(2-s, \chi) = I,$$

and

$$\varepsilon(u, s, \chi) = \Phi(s)\varepsilon(u, 2-s, \chi).$$

Furthermore $\varepsilon(u, s, \chi)$ is meromorphic with respect to s on the whole complex plane, and the poles of E_i are simple and independent of u, i. e., if s_0 is a pole of E_i, then $(s - s_0)E_i$ has no singularity with respect to u and s, whenever s is in a suitable neighbourhood of s_0[20].

20) For the general theory of Eisenstein series, see [17], [18] and [12]. The functioanal equation is not used in this paper.

§ 6. Unitary representations determined by Eisenstein series.

In the preceding §, we have introduced Eisenstein series $E_i(u, s, \chi)$, $(i = 1, 2, \ldots, h)$, h being the number of essential cusps with respect to the charcter χ of the discontinuous group Γ of Hilbert' s type operating on the direct product H^r of the upper half space H. The functions E_i are automorphic with respect to u in the sense that they satisfy (26), but, for general s, they are not square integrable on a fundamental domain $\Gamma \backslash H^r$ of Γ. Our aim is now to construct a certain finite dimensional space over $\mathbf{C}$ of automorphic functions which are derived from Eisenstein series and are square integrable on $\Gamma \backslash H^r$. This is done by investigating E_i at the point $s = (n+1)/n$. To speak precisely, denote by $\varepsilon_{s,\chi}$ the space of all functions of the form $\sum_{i=1}^{h} w_i(s) E_i(u, s, \chi)$, where w_i is a holomorphic function on the whole s-plane. Furthermore, denote by Θ_χ the space of all functions $\theta(u, \chi)$ on H^r which are square integrable on $\Gamma \backslash H^r$ and are expressed in the form

$$\theta(u, \chi) = \lim_{s \to (n+1)/n} f(u, s, \chi), \qquad f(u, s, \chi) \in \varepsilon_{s,\chi}.$$

The space Θ_χ is then finite dimensional over $\mathbf{C}$, every $\theta(u, \chi)$ is an eigenfunction of all Laplacians on H^r, and satisfies

$$(44) \qquad \theta(\sigma u, \chi) = \chi(\sigma)\theta(u, \chi), \qquad (\sigma \in \Gamma),$$

i. e., $\theta(u, \chi)$ is automorphic. Making use of the space Θ_χ, we shall obtain later a special kind of unitary representation of the metaplectic group $\tilde{G}_A$. The non-triviality of Θ_χ is assured by the following

Theorem 6. If the natural number N which is used in the definition of the compact subgroup K of the adele group G_A is divisible by n^2, and is an n-th power in F, then the space Θ_χ is not empty.

Proof. Since ∞ is always an essential cusp, one can find among $E_i(u, s, \chi)$, $(i = 1, 2, \ldots, h)$, a series corresponding to ∞. We denote it simply by $E(u, s, \chi)$ and propose to show that

$$(45) \qquad \theta(u, \chi) = \lim_{s \to (n+1)/n} \left(s - \frac{n+1}{n}\right) E(u, s, \chi)$$

actually belongs to Θ_χ. Of course, this is enough to prove the theorem.

We now investigate the Fourier expansion of E at every essential cusp $\varkappa_j$. For the sake of simplicity, let us abbreviate i in all notations concerning the Fourier expansion. In particular, let us write φ_j, χ_j, or M_j for φ_{ij}, χ_{ij}, or M_{ij} in (40) and (36).
Furthermore, for $\varkappa_j = \infty$, let us also abbreviate j, and write simply φ, χ, or M. We see then

$$\sum_{\substack{(c,\, d) \in M \\ d \bmod c\mathfrak{m}}} \bar{\chi}(c, d) = \sum_{\substack{c \equiv d - 1 \equiv (N) \\ d \text{ prime to } c \\ d \bmod c\mathfrak{m}}} \left(\frac{c}{d}\right)^{-1} = \|\mathfrak{m}\| \sum_{\substack{d \equiv 1 (N) \\ d \text{ prime to } c_0 \\ d \bmod c_0 N}} \left(\frac{c}{d}\right)^{-1}$$

$$= \begin{cases} \|\mathfrak{m}\| \varphi_F(c_0 N)\varphi_F(N)^{-1}, & \text{if } c_0 \text{ is an n-th power modulo certain finitely generated group of numbers,} \\ 0, & \text{otherwise,} \end{cases}$$

where $c = c_0 N \in \mathfrak{C}$, and φ_F means Euler's function in F. If c_0 is an n-th power, we have $(c_0) = (c_1)^n$ and $N = \nu^n$ with $c_1, \nu \in \mathfrak{o}$, and $\varphi_F(c_0 N) = \varphi_F(c_1 \nu)\|c_1 \nu\|^{n-1}$. Therefore, the Dirichlet series contained in (40) is up to a constant factor equal to

$$(46) \qquad \sum_{(c_1) \subset \mathfrak{o}} \frac{\varphi_F(c_1 \nu)}{\|c_1 \nu\|^{ns-(n-1)}} = \|N\|^{-s} \sum_{(c_1)} \frac{\varphi_F(c_1 \nu)}{\|c_1\|^{ns-(n-1)}} .$$

We want to prove that the function of s determined by the series (46) has a pole of first order at $s = (n+1)/n$. For this purpose, it is enough to show that $\sum_{(c_1)} \varphi_F(c_1)/\|c_1\|^{ns-(n-1)}$ has the same property, because (46) is convergent for $\mathrm{Re}\, s > (n+1)/n$, and because we have $\varphi_F(c) \leqq \varphi_F(c\nu) \leqq \varphi_F(c)\|\nu\|$[21]. Let Λ be a character of the absolute ideal class of F, then

$$\sum_{\mathfrak{a}} \frac{\Lambda(\mathfrak{a})\varphi_F(\mathfrak{a})}{\|\mathfrak{a}\|^{ns-(n-1)}} \cdot \sum_{\mathfrak{b}} \frac{\Lambda(\mathfrak{b})}{\|\mathfrak{b}\|^{ns-(n-1)}} = \sum_{\mathfrak{c}} \frac{\Lambda(\mathfrak{c})\|\mathfrak{c}\|}{\|\mathfrak{c}\|^{ns-(n-1)}},$$

21) It is a consequence of the general theory that all functions which we concern here are meromorphic.

$\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ running over all integral ideals of F. Hence

$$\sum_{\mathfrak{a}} \frac{\Lambda(\mathfrak{a})\varphi_F(\mathfrak{a})}{\|\mathfrak{a}\|^{ns-(n-1)}} = L(ns-n, \Lambda)/L(ns-(n-1), \Lambda)$$

with Hecke's L-series $L(s, \Lambda)$. At $s = (n+1)/n$, $L(ns-(n-1), \Lambda)$ is holomorphic and not equal to 0, while $L(ns-n, \chi)$ is holomorphic for non-trivial Λ, and has a pole of first order for trivial Λ. From this follows immediately our assertion, and at the same time one sees that $\varphi(s, \chi)$ has a pole of first order at $s = (n+1)/n$.

It follows now from the general theory of the Eisenstein series that the singularities of $E(u, s, \chi)$ in the half s-plane $\mathrm{Re}\, s > 1$ are singularities of $\varphi(s, \chi)$, and even at these singularities $E(u, s, \chi)/\varphi(s, \chi)$ is holomorphic with respect to s[22]. Thus, we obtain by (45) an automorphic form $\theta(u, \chi)$ satisfying (44), and (39) implies that the constant term of the Fourier expansion of $\theta(u, \chi)$ at every essential cusp x_j is, up to a constant factor, of the form $v(u)^{(n-1)/n}$. It follows here again from the general theory of Eisenstein series that $\theta(u, \chi)$ is bounded on a domain $\mathcal{D}'$ which is obtained from a fundamental domain $\mathcal{D} = \Gamma\backslash H^r$ by removing arbitrarily small neighbourhoods of all essential cusps ; furthermore it follows that the sum of all non-constant terms of the Fourier expansion of $\theta(u, \chi)$ at an essential

22) The idea of a proof of this result is described in [18]. Details of the proof are in [12]. In the present case, however, the key point of the proof of our assertion is the same evaluation as mentioned in § 4, 2. of [11]. Cf. also the seminar note of the author; Elementary theory of the Eisenstein series, (in Japanese), Tokyo University 1968.

cusp κ_j is bounded in a neighbourhood of ∞. On the other hand, if $\mathfrak{D}_Y$ denotes a subdomain determined by $v(u) > Y$, $(Y > 0)$, of a fundamental domain of a group as Γ_0 in Prop. 9, then $v(u)^{(n-1)/n}$ is square integrable in $\mathfrak{D}_Y$. Therefore, $\theta(u, \chi)$ itself is square integrable in $\mathfrak{D}$, and Θ_χ contains consequently the non-trivial function $\theta(u, \chi)$, which completes the proof.

Our final task is to verify that Θ_χ constitutes a representation module of the Hecke algebra. Of course, the correspondence $f \leftrightarrow f_A$ given by (24) enables us to regard $\varepsilon_{s,\chi}$ and Θ_χ as spaces of functions on $\tilde{G} = \hat{G}_F K_0 G_\infty$, and under this situation we investigate the action of the Hecke operator $æ_\chi(\tilde{G}, K\mathfrak{z})$ on $\varepsilon_{s,\chi}$ and Θ_χ.

For the Eisenstein series $E_i(u, s, \chi)$, denote by $E_{A,i}(g, s, \chi)$ the corresponding function on $\tilde{G}$ in the above sense, and put

$$v_A^s(g) = \begin{cases} v(g_\infty)^s \zeta^{-1}, & \text{if } g \in K_0 G_\infty \cdot \mathring{\zeta}, \\ 0, & \text{otherwise,} \end{cases}$$

where $v(g_\infty)$ means the value of the function $v(u)$ at the point $u \in H^r$ which is determined naturally by $g_\infty \in GL(2, \mathbf{C})^r$ through the relation $H^r = GL(2, \mathbf{C})^r / \Delta K_\infty$. Then we have

Proposition 10. The meanings of $\sigma_i \in SL(2, F)$ and Γ_i be as in the definition (34) of the Eisenstein series, choose the group Γ_0 in Prop. 9 in such a way that Γ_0 is contained in all $\sigma_i^{-1}\Gamma_i\sigma_i$, $(i = 1, 2, \ldots, h)$. Furthermore, put $\tilde{a} = (a, 1) \in \tilde{G}_A$ for any $a \in G_A$, and put

$\sigma_i\Gamma_0\sigma_i^{-1} = \Gamma_i^!$. Then, $E_{A,i}(g, s, \chi)$ is equal to

$$(47) \qquad \sum_{\sigma \in \hat{\Gamma}_i^! \backslash \hat{G}_F} v_A^s((\tilde{\sigma}_i)_\infty^{-1}\sigma g)$$

up to a constant factor.

Proof. By Prop. 9, $\sigma_i\Gamma_0\sigma_i^{-1}$ is a subgroup of a finite index ℓ of Γ_i. Therefore, (47) is equal to $\ell \cdot \sum_{\sigma \in \Gamma_i \backslash \Gamma} \overline{\chi}(\sigma)v(\sigma_i^{-1}\sigma u)^s$, which proves the proposition.

Theorem 7. For any $\psi \in æ_\chi(\tilde{G}, K\mathfrak{z})$, we have $\varepsilon_{s,\chi}^{T(\psi)} \subset \varepsilon_{s,\chi}$.

Proof. In the beginning of § 4, we have notified the decomposition $æ_\chi(\tilde{G}, K\mathfrak{z}) = æ(G_\infty, K_\infty) \otimes æ_{\chi,0}(\tilde{G}, K\mathfrak{z})$. Since $E_i(u, s, \chi)$ is an eigenfuction of Laplacians on H^r, we see that $E_{A,i}^{T(\psi)}$ for $\psi \in æ(G_\infty, K_\infty)$ is a multiple of E_i by a holomorphic function of s on the whole s-plane. So, for the proof of the theorem, it is enough to show that $E_{A,i}^{T(\psi)}$ belongs to $\varepsilon_{s,\chi}$ for $\psi \in æ_{\chi,0}(\tilde{G}, Kz)$. If we put $G_{0,I} = G_0 \cap G$ with $G = G_FK_0G_\infty$, $\tilde{G}_{0,I}$ the inverse image of the covering map $\tilde{G}_A \to G_A$, then $æ_{\chi,0}(\tilde{G}, Kz) = æ_\chi(\tilde{G}_{0,I}, K_0z)$, and what we have to show turns out

$$(48) \qquad \int_{\tilde{G}_{0,I}} E_{A,i}(gy^{-1}, s, \chi)\psi_\chi(y)\,dy \in \varepsilon_{s,\chi}$$

for any continuous function ψ with compact support on $\tilde{G}_{0,\, I}$ satisfying $\psi(y\dot{\zeta}) = \psi(y)\zeta^{-1}$. The integral in (48) is not the proper convolution on $\tilde{G}_{0,\, I}$. Since, however, no confusion is possible, we use temporarily in this proof the notation $*$ to indicate a rather modified convolution so that (48) may be written as $E_{A,\, i} * \psi \in \varepsilon_{s,\, \chi}$.

The function $v_A^s((\tilde{\sigma}_i)_\infty^{-1} g) * \psi$ is a finite linear combination of functions of the form $v_A^s((\tilde{\sigma}_i)_\infty^{-1}(\tilde{\beta})_0 g)$, $(\beta \in G_F)$, and v_A^s satisfies $v_A^s((\tilde{\gamma})_0 g) = v_A^s(g)$ for any $\gamma \in \Gamma \subset G_F$. Therefore, it follows from Prop. 10 that $E_{A,\, i} * \psi$ is up to a constant factor equal to a linear combination of $\sum_{\hat{\Gamma}_i' \backslash \hat{G}_F} v_A^s((\tilde{\sigma}_i)_\infty^{-1}(\tilde{\beta})_0 \sigma g)$. Now, making $\sigma_i^{-1}\beta^{-1}$ operate on H^r, and considering in particular the point $\varkappa \in F \subset \mathbf{C}^r$ with $\sigma_i^{-1}\beta^{-1}\varkappa = \infty$, one sees that there exist a $\gamma \in \Gamma$, $\tau = \begin{pmatrix} a' & b' \\ & c' \end{pmatrix} \in GL(2, F)$, and a σ_j among $\sigma_1, \ldots, \sigma_h$ such that $\sigma_i^{-1}\beta^{-1} = \tau\sigma_j^{-1}\gamma$. So, putting

$$\gamma\beta\sigma_i\Gamma_0\sigma_i^{-1}\beta^{-1}\gamma^{-1} = \sigma_j\tau^{-1}\Gamma_0\tau\sigma_j^{-1} = \Gamma_j'',$$

and using the fact $v_A^s((\tau)_\infty g) = \|a'/c'\|^s v_A^s(g)$, we have

$$\sum_{\sigma \in \hat{\Gamma}_i' \backslash \hat{G}_F} v_A^s((\tilde{\sigma}_i)_\infty^{-1}(\tilde{\beta})_0 \sigma g) = \sum_{\hat{\Gamma}_i' \backslash \hat{G}_F} v_A^s((\tilde{\tau})_\infty(\tilde{\sigma}_j)_\infty^{-1}(\tilde{\gamma})_\infty \tilde{\beta}\sigma g)$$

$$= \|a'/c'\|^s \sum_{\hat{\Gamma}_i' \backslash \hat{G}_F} v_A^s((\tilde{\gamma})_0^{-1}(\tilde{\sigma}_j)_\infty^{-1}\tilde{\gamma}\tilde{\beta}\sigma g) = \|a'/c'\|^s \sum_{\hat{\Gamma}_j'' \backslash \hat{G}_F} v_A^s((\tilde{\sigma}_j)_\infty^{-1}\sigma g).$$

Since the last sum is up to a constant factor equal to $E_{A,\,j}$ by Prop. 10, the theorem is proved.

From Theorem 7 it follows that the space of all functions of the form $f(g,\ n+1/n)$, $(f(g,\ s) \in \varepsilon_{s,\,\chi})$, is mapped into itself by any $T(\psi)$ with $\psi \in æ_{\chi}(\tilde{G},\ K\mathfrak{z})$. On the other hand, the image by $T(\psi)$ of a square integrable function on $\hat{G}_F \backslash \tilde{G}$ is again square integrable. Since Θ_{χ} is the space of all $f(g,\ n+1/n)$ which are square integrable on $\hat{G}_F \backslash \tilde{G}$, one obtains the following

Theorem 8. For any $\psi \in æ_{\chi}(\tilde{G},\ K\mathfrak{z})$, we have $\Theta_{\chi}^{T(\psi)} \subset \Theta_{\chi}$.

The space Θ_{χ} is finite dimensional over $\mathbf{C}$, and consists of functions $\theta_A(x)$ on $\tilde{G}$ satisfying $\theta_A(\gamma x) = \theta_A(x)$, $(\gamma \in \hat{G}_F)$, and $\theta_A(x\dot{\zeta}) = \theta_A(x)\zeta^{-1}$, $(\dot{\zeta} \in \mathfrak{z})$. Hence, by Theorem 8 and by the generalities of the unitary representation explained in § 2, we have

Theorem 9. Let $\mathfrak{H}$ be the Hilbert space, with the inner product $(f_1,\ f_2) = \int_{\hat{G}_F \backslash \tilde{G}} f_1(x)\overline{f_2(x)}dx$, genetated by all functions of $x \in \tilde{G}$ of the form $\theta_A(xg)$, $(\theta_A \in \Theta_{\chi},\ g \in \tilde{G})$, then the unitary representation $\{U_g\}$, $(g \in \tilde{G})$, determined by $(U_g f)(x) = f(xg)$ .is decomposed into a sum of finite irreducible unitary representations of $\tilde{G}$. Furthermore, $U_{\dot{\zeta}}$ is the multiplication by ζ^{-1}, if $\dot{\zeta} \in \mathfrak{z}$.

Theorem 9 is the main aim of the present paper. Using the method of ordinary induced representations, it is no longer hard to construct a representation of $\tilde{G}_A$ which has all corresponding properties of the representation of $\tilde{G}$ in Theorem 9; for, as was mentioned in § 1, $\tilde{G}$ is a normal subgroup of finite index of $\tilde{G}_A$.

§ 7. Further miscellaneous results and remarks.

1. Let $æ'_{\chi,0}(\widetilde{G}, K\mathfrak{z})$ be the algebra introduced in the beginning of § 4. Then, by Theorem 4, $æ'_{\chi,0}(\widetilde{G}, K\mathfrak{z})$ is commutative, and is a subalgebra of $æ_{\chi,0}(\widetilde{G}, K\mathfrak{z})$. On the other hand, it follows from the argument in § 2 about general Hecke operators that the adjoint operator of $T(\psi)$, $\psi \in æ'_{\chi,0}(\widetilde{G}, K\mathfrak{z})$, is again a Hecke operator of the same type. So, Theorem 8 implies that $æ'_{\chi,0}(\widetilde{G}, K\mathfrak{z})$ is represented by mutually commutative normal operators of Θ_{χ}, and consequently Θ_{χ} is a derect sum of one dimensional subspaces over $\mathbf{C}$ which are $æ'_{\chi,0}(\widetilde{G}, K\mathfrak{z})$ modules. If $\mathbf{C}\theta$ is one of such subspaces, then $\theta^{T(\psi)} = \theta * \psi = \rho_{\psi}\theta$, $(\rho_{\psi} \in \mathbf{C})$, for any $\psi \in æ'_{\chi,0}(\widetilde{G}, K\mathfrak{z})$. We now propose to show that the eigenvalue ρ_{ψ} of the Hecke operator $T(\psi)$ is of a very simple nature. Namely, at least under certain conditions which do not essentially restrict the generality ρ_{ψ} is an elementary arithmetical sum similar to the power sum of divisors of an integer. The latter is indeed the eigenvalue of the classical analytic Eisenstein series under the action of Hecke operators. We assume first that the constant term of the Fourier expansion of θ at ∞ is not 0. Such a θ actually exists, because Theorem 6 is proved by finding a function which satisfy the same condition at ∞. Next we assume that the Hecke operator $T(\psi)$ is the operator $T(1, \omega^{nt})$ in (29), (30). For the sake of convenience, regard now Θ_{χ} as a space of functions on H^{r}, and apply (30). Then it follows from (38) that ρ_{ψ} is obtained by observing the effect of the Hecke operator only on the constant term of the Fourier expansion of θ at ∞. The constant term is $v(u)^{(n-1)/n}$

up to a constant factor, and $\sum_{c \bmod \omega^k,\ \omega \nmid c} (\frac{c}{\omega})^k = 0$ if k is not divisible by n. So, denoting ρ_ψ by $\rho(1, \omega^{nt})$ in this special case, we have

$$(49) \qquad (q^{nt} + q^{nt-1}) \cdot \rho(1, \omega^{nt}) = q^{\frac{1}{2}(n-1)t}(1 + \sum_{\ell=1}^{t-1} q^{-(n-1)\ell} \cdot (q^{n\ell} - q^{n\ell-1}) + q^{-(n-1)t} q^{nt}).$$

Here, the term with $\sum$ must be removed if $t = 1$.

If we observe a suitable subgroup of $\tilde{G}$, for instance the product $G' = G_\infty \times (\widetilde{G'_0} \cap \tilde{G})$, $\widetilde{G'_0}$ being the inverse image with respect to the covering map $\tilde{G}_A \to G_A$ of the group G'_0 in Prop. 7, then the function $\omega(g)$ of $g \in G'$ given by $\|\theta\|^2 \omega(g) = (\theta, U_g\theta) = \int_{\hat{G}_F \backslash \tilde{G}} \theta(x)\overline{\theta(xg)}dx$, $(g \in G')$, is the zonal spherical function of the irreducible unitary representation of G' determined by unitary operators U_g; $f(x) \to f(xg)$ of the Hilbert space $\mathfrak{H}_\theta$ spanned by $\theta(xg)$, $g \in G'$. If $\int_K \theta(xk^{-1}g)\chi(k)dk = \rho_g\theta(x)$, then we have $(\theta, U_g\theta) = \bar{\rho}_g$. This result which means that the zonal spherical function is the same thing as the eigenvalue of the Hecke operator in the sense we have introduced in §2, and which is of course merely an elementary fact in the general theory of the unitary representation, shows, however, together with (49) that the zonal spherical function $\omega(g)$ of the unitary representation of G' by H_θ is, at least under some restriction of $g \in G'$, given by a simple sum as (49).

2. In [22] it was shown that a special type of unitary representation of SL(2) over a local field F is obtained by the representation of the type discovered in [23] of the metaplectic group over any quadratic extension F' of F, and this result is obtained by using a natural imbedding of SL(2, F) in the metaplectic group. We now propose to explain that a similar situation is found in a rather generalized form also in the case of our $\tilde{G}_A$. Let F' be an extension of degree n over a totally imaginary algebraic number field F containing the n-th roots of unity, and let $\tilde{G}_A$ be the (generalized) metaplectic group over F' in the sense of § 3. Then, it follows from the properties of norm residue symbol[23] and from the definition that $\prod_{\mathfrak{q}|\mathfrak{p}} a_{\mathfrak{q}}(g, g') = 1$ for any place $\mathfrak{p}$ of F and $\prod_{\mathfrak{q}|\mathfrak{p}} s_{\mathfrak{q}}(g) = 1$ for any $\mathfrak{p}$ of F which is not ramified in F'[24], the product being extended over the places $\mathfrak{q}|\mathfrak{p}$ of F' in both cases. Thus the factor set $b_A(g, g')$ splits if g, g' are restricted to adeles of GL(2, F). Hence, $\tilde{G}_A$ contains a group which is isomorphic to the adele group $GL_A(2)$ of the general linear group of degree 2 over F, and the representation obtained by Theorem 9 induces a representation of $GL_A(2)$.

3. For a point $u = \begin{pmatrix} z & -v \\ v & \bar{z} \end{pmatrix}$ of the upper half space H, put $I(u) = \begin{pmatrix} z & iv \\ iv & \bar{z} \end{pmatrix}$, and for $u = (u_1, \ldots, u_r) \in H^r$ as in (5), put

23) See in particular the formula (9.) in p. 54 of [4].
24) If F'/F is normal, then the second equality holds also for any p.

$$I(u) = \begin{pmatrix} I(u_1) & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & & I(u_r) \end{pmatrix}.$$

Furthermore, let $\alpha_1, \ldots, \alpha_{2r}$ be a $\mathbf{Z}$-basis of an ideal $\mathfrak{a}$ of F, and put

$$A = \begin{pmatrix} \alpha_1^{(1)} & \alpha_2^{(1)} & \cdots\cdots & \alpha_{2r}^{(1)} \\ \bar{\alpha}_1^{(1)} & \bar{\alpha}_2^{(1)} & \cdots\cdots & \alpha_{2r}^{(1)} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \alpha_1^{(r)} & \alpha_2^{(r)} & \cdots\cdots & \alpha_{2r}^{(r)} \\ \bar{\alpha}_1^{(r)} & \bar{\alpha}_2^{(r)} & \cdots\cdots & \bar{\alpha}_{2r}^{(r)} \end{pmatrix};$$

A is the matrix of complete conjugates of $\alpha_1, \ldots, \alpha_{2r}$ over $\mathbf{Q}$.
Then, ${}^tAI(u)A$ is a point of the Siegel's space $\mathfrak{H}_r$ of degree r, and the theta function $\sum_{\mathfrak{x} \in \mathbf{Z}^n} \exp(\pi\sqrt{-1}\,{}^t\mathfrak{x}Z\mathfrak{x})$, $(Z \in \mathfrak{H}_r)$, determines a function $\vartheta(u)$ of $u \in H^r$ through

$$(50) \qquad \vartheta(u) = \vartheta({}^tAI(u)A)$$

$$= \sum_{\alpha \in \mathfrak{a}} \exp(\pi\sqrt{-1} \sum_{\ell} (\alpha^{(\ell)2} z_\ell + 2\sqrt{-1}\,|\alpha^{(\ell)}|^2 v_\ell + \bar{\alpha}^{(\ell)2} \bar{z}_\ell))^{25)}.$$

If now $n = 2$, then the Dirichlet series (42) becomes essentially the

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25) Cf. [8].

Hecke's L-series with a quadratic class character, and using for example the method as in [13] or [20] one can show that the space Θ_χ in § 6 is a space of functions like (50). This fact suggests that our representation in Theorem 9 is of a similar nature to the representation in [23]. But, satisfactory results for $n > 2$ are not obtained yet, and for this purpose it seems to be necessary to investigate the representation in Theorem 9 for each local component $\widetilde{G}_{\mathfrak{p}}$ of $\widetilde{G}_A$.

4. The Dirichlet series (42) is, in case $n > 2$, expressed by special kind of zeta functions as introduced in [11]. So, the representation in Theorem 9 is, so to speak, constructed by using the values, or residues, of such special zeta functions at $s = (n+1)/n$.

5. As in the classical case, linear relations between coefficients of Fourier series (38) for an eigenfunction of Hecke operators are obtained by means of the explicit expression (30) of Hecke operators. For $n = 2$, this kind of investigation was done in [24].

The operation of (30) on the exponential function $e(m, u)$ in (38) yields clearly a Gauss sum containing a congruence character of order n. Therefore, the nature of the case of $n > 2$ becomes largely different from the case of $n = 2$.

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